Chapter 3
The Z Transform

3.8 Z-Transform Inversion Techniques

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February 12, 2008
Z-Transform Inversion Techniques

- The most fundamental method for the inversion of a z transform is the *general inversion method* which is based on the Laurent theorem.
The most fundamental method for the inversion of a $z$ transform is the *general inversion method* which is based on the Laurent theorem.

In this method, the inverse of a $z$ transform $X(z)$ is given by

$$x(nT) = \frac{1}{2\pi j} \oint_{\Gamma} X(z) z^{n-1} \, dz$$

where $\Gamma$ is a closed contour in the counterclockwise sense enclosing all the singularities of function $X(z)z^{n-1}$. 
\[ x(nT) = \frac{1}{2\pi j} \oint_{\Gamma} X(z) z^{n-1} \, dz \]

At first sight, the above contour integration may appear to be a formidable task.
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- At first sight, the above contour integration may appear to be a formidable task.

- However, for most DSP applications, the \( z \) transform turns out to be a rational function and for such functions the contour integral can be easily evaluated by using the residue theorem.
According to the residue theorem,

\[
x(nT) = \frac{1}{2\pi j} \oint_{\Gamma} X(z)z^{n-1} \, dz = \sum_{i=1}^{P} \text{Res}_{z \to p_i} [X(z)z^{n-1}]
\]

where \( \text{Res}_{z \to p_i} [X(z)z^{n-1}] \) and \( P \) are the residue of pole \( p_i \) and the number of poles of \( X(z)z^{n-1} \), respectively.
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For a pole of order \( m_i \),

\[ \text{Res}_{z=p_i} \left[ X(z)z^{n-1} \right] = \frac{1}{(m_i - 1)!} \lim_{z \to p_i} \frac{d^{m_i-1}}{dz^{m_i-1}} \left[ (z - p_i)^{m_i} X(z)z^{n-1} \right] \]
According to the residue theorem,

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For a pole of order $m_i$,

$$\text{Res}_{z=p_i} [X(z)z^{n-1}] = \frac{1}{(m_i - 1)!} \lim_{z\to p_i} \frac{d^{m_i-1}}{dz^{m_i-1}} \left[(z - p_i)^{m_i} X(z)z^{n-1}\right]$$

For a simple pole,

$$\text{Res}_{z=p_i} [X(z)z^{n-1}] = \lim_{z\to p_i} \left[(z - p_i)X(z)z^{n-1}\right]$$
Example – General Inversion Method

Using the general inversion method, find the inverse $z$ transform of

$$X(z) = \frac{1}{2(z - 1)(z + \frac{1}{2})}$$

**Solution** We note that the factor $z^{n-1}$ introduces a pole in $X(z)z^{n-1}$ at the origin for the case $n = 0$, which must be taken into account in the evaluation of $x(0)$.

*Note:* For $n > 0$, the pole at the origin *disappears.*
Thus for $n = 0$, we have

$$X(z)z^{n-1} \bigg|_{n=0} = \frac{z^{n-1}}{2(z - 1)(z + \frac{1}{2})} \bigg|_{n=0} = \frac{1}{2z(z - 1)(z + \frac{1}{2})}$$

Hence

$$x(0) = \left. \frac{1}{2(z - 1)(z + \frac{1}{2})} \right|_{z=0} + \left. \frac{1}{2z(z + \frac{1}{2})} \right|_{z=1} + \left. \frac{1}{2z(z - 1)} \right|_{z=-\frac{1}{2}} = -1 + \frac{1}{3} + \frac{2}{3} = 0$$

Actually, this follows from the initial-value theorem (Theorem 3.8) without any calculations.
For $n > 0$

\[
x(nT) = \frac{z^{n-1}}{2(z + \frac{1}{2})} \bigg|_{z=1} + \frac{z^{n-1}}{2(z - 1)} \bigg|_{z=-\frac{1}{2}}
\]

\[
= \frac{1}{3} - \frac{1}{3} \left(-\frac{1}{2}\right)^{n-1}
\]

and from the initial-value theorem, $x(nT) = 0$ for $n < 0$.

Therefore, for any value of $n$, we have

\[
x(nT) = u(nT - T) \left[\frac{1}{3} - \frac{1}{3} \left(-\frac{1}{2}\right)^{n-1}\right]
\]
Example – General Inversion Method

Using the general inversion method, find the inverse $z$ transform of

$$X(z) = \frac{(2z - 1)z}{2(z - 1)(z + \frac{1}{2})}$$

**Solution** We can write

$$X(z)z^{n-1} = \frac{(2z - 1)z \cdot z^{n-1}}{2(z - 1)(z + \frac{1}{2})} = \frac{(2z - 1)z^n}{2(z - 1)(z + \frac{1}{2})}$$

We note that $X(z)z^{n-1}$ has simple poles at $z = 1$ and $-\frac{1}{2}$.

Furthermore, the zero in $X(z)$ at the origin cancels the pole at the origin introduced by $z^{n-1}$ for the case $n = 0$. 


\[ X(z)z^{n-1} = \frac{(2z - 1)z^n}{2(z - 1)(z + \frac{1}{2})} \]

Hence for any \( n \geq 0 \), the general inversion formula gives

\[ x(nT) = \text{Res}_{z=1} [X(z)z^{n-1}] + \text{Res}_{z=-\frac{1}{2}} [X(z)z^{n-1}] \]

\[ = \frac{(2z - 1)z^n}{2(z + \frac{1}{2})} \bigg|_{z=1} + \frac{(2z - 1)z^n}{2(z - 1)} \bigg|_{z=-\frac{1}{2}} \]

\[ = \frac{1}{3} + \frac{2}{3} \left(-\frac{1}{2}\right)^n \]
Since the numerator degree in $X(z)$ does not exceed the denominator degree, it follows that $x(nT)$ is a right-sided signal, i.e., $x(nT) = 0$ for $n < 0$, according to the Corollary of Theorem 3.8.

Therefore, for any value of $n$, we have

$$x(nT) = u(nT) \left[ \frac{1}{3} + \frac{2}{3} \left( -\frac{1}{2} \right)^n \right]$$

where $u(nT)$ is the unit-step function.
Since

– the z transform is a particular type of Laurent series, and
– the Laurent series in a given annulus of convergence is unique

it follows that *any technique* that can be used to generate a power series for $X(z)$ that converges in the outermost annulus of convergence can be used to obtain the inverse $z$ transform.
Consequently, several inversion techniques are available, as follows:

- using the binomial theorem,
- using the convolution theorem,
- performing long division,
- using the initial-value theorem, or
- expanding $X(z)$ into partial fractions.

Some of these techniques are illustrated by examples in the next few slides.
Using the binomial method, find the inverse $z$ transform of

$$X(z) = \frac{Kz^m}{(z - w)^k}$$

where $m$ and $k$ are integers, and $K$ and $w$ are constants, possibly complex.

**Solution** The inverse $z$ transform can be obtained by obtaining a binomial series for $X(z)$ that converges in the outside annulus of $X(z)$. 
Such a binomial series can be obtained by expressing $X(z)$ as

$$X(z) = Kz^{m-k}[1 + (-wz^{-1})]^{-k}$$

$$= Kz^{m-k} \left[ (-k) + \binom{-k}{1}(-wz^{-1}) + \binom{-k}{2}(-wz^{-1})^2 + \cdots + \binom{-k}{n}(-wz^{-1})^n + \cdots \right]$$

where

$$\binom{-k}{n} = \frac{(-k)(-k - 1) \cdots (-k - n + 1)}{n!}$$

Hence

$$X(z) = \sum_{n=-\infty}^{\infty} Ku(nT) \frac{(-k)(-k - 1) \cdots (-k - n + 1)(-w)^nz^{-n+m-k}}{n!}$$
Example  Cont’d

\[
X(z) = \sum_{n=-\infty}^{\infty} Ku(nT) \frac{(-k)(-k - 1) \cdots (-k - n + 1)(-w)^n z^{-n+m-k}}{n!}
\]

Now if we let \( n = n' + m - k \) and then replace \( n' \) by \( n \), we have

\[
X(z) = \sum_{n=-\infty}^{\infty} \left\{ Ku[(n + m - k) T] \right\} \frac{(-k)(-k - 1) \cdots (-n - m + 1)(-w)^{n+m-k}}{(n + m - k)!} z^{-n}
\]
Example Cont’d

\[ X(z) = \sum_{n=-\infty}^{\infty} \left\{ Ku[(n + m - k)T] \right. \]
\[ \left. \times \frac{(-k)(-k-1) \cdots (-n - m + 1)(-w)^{n+m-k}}{(n + m - k)!} \right\} z^{-n} \]

Hence the coefficient of \( z^{-n} \) is obtained as

\[ x(nT) = Z^{-1} \left[ \frac{Kz^m}{(z - w)^k} \right] \]
\[ = Ku[(n + m - k)T] \frac{(-k)(-k-1) \cdots (-n - m + 1)(-w)^{n+m-k}}{(n + m - k)!} \]

By assigning different values to constants \( k, K, \) and \( m \) a variety of \( z \)-transform pairs can be deduced as shown in the next slide.
### Example Cont’d

<table>
<thead>
<tr>
<th>$x(nT)$</th>
<th>$X(z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u(nT)$</td>
<td>$\frac{z}{z-1}$</td>
</tr>
<tr>
<td>$u(nT - kT)K$</td>
<td>$\frac{Kz^{-(k-1)}}{z-1}$</td>
</tr>
<tr>
<td>$u(nT)Kw^n$</td>
<td>$\frac{Kz}{z-w}$</td>
</tr>
<tr>
<td>$u(nT - kT)Kw^{n-1}$</td>
<td>$\frac{K(z/w)^{-(k-1)}}{z-w}$</td>
</tr>
<tr>
<td>$u(nT)e^{-\alpha nT}$</td>
<td>$\frac{z}{z - e^{-\alpha T}}$</td>
</tr>
<tr>
<td>$r(nT)$</td>
<td>$\frac{Tz}{(z-1)^2}$</td>
</tr>
<tr>
<td>$r(nT)e^{-\alpha nT}$</td>
<td>$\frac{Te^{-\alpha T}z}{(z - e^{-\alpha T})^2}$</td>
</tr>
</tbody>
</table>
Use of Real Convolution

- From the real-convolution theorem

\[ Z \sum_{k=-\infty}^{\infty} x_1(kT)x_2(nT - kT) = X_1(z)X_2(z) \]
**Use of Real Convolution**

- From the real-convolution theorem

\[
\mathcal{Z} \sum_{k=-\infty}^{\infty} x_1(kT)x_2(nT - kT) = X_1(z)X_2(z)
\]

- If we take the inverse \( z \) transform of both sides, we get

\[
\sum_{k=-\infty}^{\infty} x_1(kT)x_2(nT - kT) = \mathcal{Z}^{-1}[X_1(z)X_2(z)]
\]

or

\[
\mathcal{Z}^{-1}[X_1(z)X_2(z)] = \sum_{k=-\infty}^{\infty} x_1(kT)x_2(nT - kT)
\]

Thus, if a \( z \) transform can be expressed as a *product* of two \( z \) transforms whose inverses are available, then performing the convolution summation will yield the desired inverse.
Example – Real Convolution

Find the inverse $z$ transform of

$$X_3(z) = \frac{z}{(z - 1)^2}$$

**Solution** We note that

$$X_3(z) = X_1(z)X_2(z)$$

where

$$X_1(z) = \frac{z}{z - 1} \quad \text{and} \quad X_2(z) = \frac{1}{z - 1}$$
Example Cont’d

\[ X_1(z) = \frac{z}{z - 1} \quad \text{and} \quad X_2(z) = \frac{1}{z - 1} \]

From the table of standard \( z \) transforms, we can write

\[ x_1(nT) = u(nT) \quad \text{and} \quad x_2(nT) = u(nT - T) \]

Hence for \( n \geq 0 \), the real convolution yields

\[
x_3(nT) = \sum_{k=-\infty}^{\infty} x_1(kT)x_2(nT - kT) = \sum_{k=-\infty}^{\infty} u(kT)u(nT - T - kT)
\]

\[
= \cdots + u(-T)u(nT) + u(0)u(nT - T) + u(T)u(nT - 2T) + \cdots
\]

\[
+ u(nT - T)u(0) + u(nT)u(-T) + \cdots
\]

\[
= 0 + 1 + 1 + \cdots + 1 + 0 = n
\]
Example  Cont’d

For \( n < 0 \), we have

\[
x_3(nT) = \sum_{k=-\infty}^{\infty} u(kT)u(nT - T - kT)
\]

\[
= \cdots + u(-T)u(nT) + u(0)u(nT - T) + u(T)u(nT - 2T) + \cdots
\]

\[
+ u(nT - T)u(0) + u(nT)u(-T) + \cdots
\]

and since all the terms are zero, we get

\[
x_3(nT) = 0
\]

(This result also follows from the initial value theorem.)
Summarizing, for $n \geq 0$, 

$$x_3(nT) = n$$

and for $n < 0$, 

$$x_3(nT) = 0$$

Therefore, for any value of $n$, we have 

$$x_3(nT) = u(nT)n$$
Example – Real Convolution

Using the real-convolution theorem, find the inverse $z$ transforms of

$$X_3(z) = \frac{z}{(z - 1)^3}$$

**Solution** For this example, we can write

$$X_1(z) = \frac{z}{(z - 1)^2} \quad \text{and} \quad X_2(z) = \frac{1}{z - 1}$$

and from the previous example, we have

$$x_1(nT) = u(nT)n \quad \text{and} \quad x_2(nT) = u(nT - T)$$
Example  Cont’d

From the initial value theorem, for \( n < 0 \), we have

\[ x_3(n) = 0 \]

For \( n \geq 0 \), the convolution summation gives

\[
x_3(nT) = \sum_{k=-\infty}^{\infty} ku(kT)u(nT - T - kT)
\]

\[
= + 0 \cdot [u(nT - T)] + 1 \cdot [u(nT - 2T)] + \cdots + (n-1)u(0) + nu(-T)
\]

\[
= + 0 + 1 + 2 + \cdots + n - 1 + 0
\]

\[
= \sum_{k=1}^{n-1} k
\]
A closed-form solution can be obtained by using an old trick of algebra.
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The story goes that Gauss’ mathematics teacher had something to attend to and wanted to keep his class busy. So he asked the class to find the sum:

\[ 1 + 2 + 3 + \cdots + 99 \]
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The story goes that Gauss’ mathematics teacher had something to attend to and wanted to keep his class busy. So he asked the class to find the sum:

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As the teacher was getting ready to leave, Gauss shouted out “Sir, the answer is 4950!”
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The story goes that Gauss’ mathematics teacher had something to attend to and wanted to keep his class busy. So he asked the class to find the sum:

\[ 1 + 2 + 3 + \cdots + 99 \]

As the teacher was getting ready to leave, Gauss shouted out “Sir, the answer is 4950!”

“It’s very simple, Sir, twice the sum is 100 \times 99”.
Gauss’ reasoning was as follows:

\[
\begin{array}{ccccccc}
1 & 2 & 3 & \cdots & n-1 \\
n-1 & n-2 & n-3 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
n & n & n & \cdots & n
\end{array}
\]

That is,

\[
\sum_{k=1}^{n-1} k = \frac{1}{2} n(n - 1)
\]

Using this result, \(x_3(nT)\) can be obtained as

\[
x_3(nT) = \sum_{k=1}^{n-1} k = \frac{1}{2} u(nT) n(n - 1)
\]
Use of Long Division

- Given a $z$ transform $X(z) = \frac{N(z)}{D(z)}$, a series that converges in the outermost annulus of $X(z)$ can be readily obtained by arranging the numerator and denominator polynomials in descending powers of $z$ and then performing polynomial division also known as *long division*. 
Using long division, find the inverse $z$ transform of

$$X(z) = \frac{-\frac{1}{4} + \frac{1}{2}z - \frac{1}{2}z^2 - \frac{7}{4}z^3 + 2z^4 + z^5}{-\frac{1}{4} + \frac{1}{4}z - z^2 + z^3}$$

**Solution** The numerator and denominator polynomials can be arranged in descending powers of $z$ as

$$X(z) = \frac{z^5 + 2z^4 - \frac{7}{4}z^3 - \frac{1}{2}z^2 + \frac{1}{2}z - \frac{1}{4}}{z^3 - z^2 + \frac{1}{4}z - \frac{1}{4}}$$
\[ \begin{align*} z^3 - z^2 + \frac{1}{4}z - \frac{1}{4} &= z^5 + 2z^4 - \frac{7}{4}z^3 - \frac{1}{2}z^2 + \frac{1}{2}z - \frac{1}{4} \\
\mp z^5 &\mp z^4 \mp \frac{1}{4}z^3 \mp \frac{1}{4}z^2 \\
3z^4 &- \frac{8}{4}z^3 - \frac{1}{4}z^2 + \frac{1}{2}z - \frac{1}{4} \\
\mp 3z^4 &\mp 3z^3 \mp \frac{3}{4}z^2 \mp \frac{3}{4}z \\
z^3 - z^2 + \frac{5}{4}z - \frac{1}{4} &= z^3 \pm z^2 \mp \frac{1}{4}z \pm \frac{1}{4} \\
\mp z &\pm 1 \mp \frac{1}{4}z^{-1} \pm \frac{1}{4}z^{-2} \\
1 - \frac{1}{4}z^{-1} + \frac{1}{4}z^{-2} &= 1 \pm z^{-1} \mp \frac{1}{4}z^{-2} \pm \frac{1}{4}z^{-3} \\
\frac{3}{4}z^{-1} &+ \frac{1}{4}z^{-3} \\
\vdots &\end{align*} \]
Therefore,

\[ X(z) = z^2 + 3z + 1 + z^{-2} + z^{-3} + \frac{3}{4}z^{-4} + \ldots \]

i.e.,

\begin{align*}
  x(-2T) &= 1, & x(-T) &= 3, & x(0) &= 1, & x(T) &= 0 \\
  x(2T) &= 1, & x(3T) &= 1, & x(4T) &= \frac{3}{4}, & \ldots
\end{align*}
As illustrated by the previous example, the long-division approach readily yields any nonzero values of the signal for $n \leq 0$ but does not yield a closed-form solution.
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On the other hand, the general-inversion method yields a closed-form solution but presents certain difficulties in $z$ transforms of two-sided signals because such $z$ transforms have a higher-order pole at the origin whose residue is difficult to obtain.
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On the other hand, the general-inversion method yields a closed-form solution but presents certain difficulties in \( z \) transforms of two-sided signals because such \( z \) transforms have a higher-order pole at the origin whose residue is difficult to obtain.

The inverses of such \( z \) transforms can be easily obtained in closed form by finding the values of the signal for \( n \leq 0 \) using long division and then applying the general inversion method to the remainder of the long division.
Consider a $z$ transform whose numerator degree exceeds the denominator degree of the form

$$X(z) = \frac{N(z)}{D(z)} = \frac{\sum_{i=0}^{M} a_i z^{M-i}}{\sum_{i=0}^{N} b_i z^{N-i}}$$
Consider a z transform whose numerator degree exceeds the denominator degree of the form

\[ X(z) = \frac{N(z)}{D(z)} = \frac{\sum_{i=0}^{M} a_i z^{M-i}}{\sum_{i=0}^{N} b_i z^{N-i}} \]

The first nonzero value of \( x(nT) \) occurs at \( n = (N - M) T \) according to the initial value theorem.
Performing long division until the signal values $x[(N - M)T]$, $x[(N - M + 1)T]$, ..., $x(0)$ are obtained, $X(z)$ can be expressed as

$$X(z) = \frac{N(z)}{D(z)} = Q(z) + R(z)$$

where

$$Q(z) = x[(N - M)T]z^{(M-N)} + x[(N - M + 1)T]z^{(M-N-1)} + \cdots + x(0)$$

is the \textit{quotient} polynomial and

$$R(z) = \frac{N'(z)}{D(z)}$$

is the \textit{remainder} whose numerator degree is less than the denominator degree.
Use of Long Division *Cont’d*

\[ X(z) = \frac{N(z)}{D(z)} = Q(z) + R(z) \quad \text{where} \quad R(z) = \frac{N'(z)}{D(z)} \]

Hence

\[ x(nT) = \mathcal{Z}^{-1}Q(z) + \mathcal{Z}^{-1} \frac{N'(z)}{D(z)} \]

\[ = x[(N - M)T]z^{(M-N)} + x[(N - M + 1)T]z^{(M-N-1)} + \ldots \]

\[ + x(0) + \mathcal{Z}^{-1} \frac{N'(z)}{D(z)} \]

Since \( \mathcal{Z}^{-1} \frac{N'(z)}{D(z)} \) represents a right-sided signal, it can be *easily* evaluated in *closed-form* by using the general inversion method.
Using long division along with the general inversion method, obtain a closed-form solution for the inverse $z$ transform of

$$X(z) = \frac{-\frac{1}{4} + \frac{1}{2}z - \frac{1}{2}z^2 - \frac{7}{4}z^3 + 2z^4 + z^5}{-\frac{1}{4} + \frac{1}{4}z - z^2 + z^3}$$
Example \ Cont’d

Solution

\[
\begin{array}{l}
z^3 - z^2 + \frac{1}{4}z - \frac{1}{4} \\
\hline
z^2 + 3z + 1 \\
\hline
z^5 + 2z^4 - \frac{7}{4}z^3 - \frac{1}{2}z^2 + \frac{1}{2}z - \frac{1}{4} \\
\underline{+ z^5 \pm z^4 \mp \frac{1}{4}z^3 \pm \frac{1}{4}z^2} \\
3z^4 - \frac{8}{4}z^3 - \frac{1}{4}z^2 + \frac{1}{2}z - \frac{1}{4} \\
\underline{+ 3z^4 \pm 3z^3 \mp \frac{3}{4}z^2 \pm \frac{3}{4}z} \\
\hline
z^3 - z^2 + \frac{5}{4}z - \frac{1}{4} \\
\underline{+ z^3 \pm z^2 \mp \frac{1}{4}z \pm \frac{1}{4}} \\
\hline
z
\end{array}
\]

Hence

\[
X(z) = Q(z) + R(z) = z^2 + 3z + 1 + \frac{z}{z^3 - z^2 + \frac{1}{4}z - \frac{1}{4}}
\]
Applying the inverse $z$ transform, we have

$$x(nT) = \mathcal{Z}^{-1}\left(z^2 + 3z + 1 + \frac{z}{z^3 - z^2 + \frac{1}{4}z - \frac{1}{4}}\right)$$

$$= x(-2T)z^2 + x(-T)z + x(0) + \mathcal{Z}^{-1}R(z)$$

where $x(-2T) = 1$, $x(-T) = 3$, $x(0) = 1$, and

$$R(z) = \frac{z}{z^3 - z^2 + \frac{1}{4}z - \frac{1}{4}} = \frac{z}{(z - 1)(z + j\frac{1}{2})(z - j\frac{1}{2})}$$

The inverse $z$ transform of $R(z)$ can now be obtained by using the general inversion method.
Example \textit{Cont'd}

\[ R(z) = \frac{Z}{z^3 - z^2 + \frac{1}{4}z - \frac{1}{4}} = \frac{Z}{(z - 1)(z + j\frac{1}{2})(z - j\frac{1}{2})} \]

Since \(-j\frac{1}{2} = \frac{1}{2}e^{-j\pi/2}\), the residues of \(R(z)z^{n-1}\) can be obtained as

\[ R_1 = \lim_{z \to 1} \frac{z^n}{(z^2 + \frac{1}{4})} = \frac{1}{1 + \frac{1}{4}} = \frac{4}{5} \]

\[ R_2 = \lim_{z \to -j\frac{1}{2}} \frac{z^n}{(z - 1)(z - j\frac{1}{2})} = \frac{\left(\frac{1}{2}\right)^n e^{-jn\pi/2}}{(-\frac{1}{2} + j)} \]

\[ = \frac{2}{\sqrt{5}} \left(\frac{1}{2}\right)^n e^{-jn\pi/2} = \frac{2}{\sqrt{5}} \left(\frac{1}{2}\right)^n e^{-j(n\pi/2 + \pi - \tan^{-1}2)} \]

\[ R_3 = R_2^* = \frac{2}{\sqrt{5}} \left(\frac{1}{2}\right)^n e^{j(n\pi/2 + \pi - \tan^{-1}2)} \]
Thus for $n \geq 1$, we have

$$R(z) = R_1 + R_2 + R_3 = \frac{4}{5} + \frac{4}{\sqrt{5}} \left(\frac{1}{2}\right)^n \left[ e^{j(n\pi/2+\pi-\tan^{-1}2)} + e^{-j(n\pi/2+\pi-\tan^{-1}2)} \right]$$

Hence

$$r(nT) = \frac{4}{5} u(nT) + \frac{4}{\sqrt{5}} \left(\frac{1}{2}\right)^n \cos(n\pi/2 + \pi - \tan^{-1}2)$$

Since $x(-2T) = 1$, $x(-T) = 3$, and $x(0) = 1$, the value of $x(nT)$ for any value of $n$ is given by

$$x(nT) = \delta(nT + 2T) + 3\delta(nT + T) + \delta(nT) + u(nT - T)[\frac{4}{5} + \frac{4}{\sqrt{5}} \left(\frac{1}{2}\right)^n \cos(n\pi/2 + \pi - \tan^{-1}2)]$$
Use of Partial Fractions

If the degree of the numerator polynomial in $X(z)$ is equal to or less than the degree of the denominator polynomial and the poles are simple, the inverse of $X(z)$ can very quickly be obtained through the use of partial fractions.
Use of Partial Fractions

- If the degree of the numerator polynomial in $X(z)$ is equal to or less than the degree of the denominator polynomial and the poles are simple, the inverse of $X(z)$ can very quickly be obtained through the use of partial fractions.

- Two techniques are available, as detailed next.
The function $X(z)/z$ can be expanded into partial fractions as

$$
\frac{X(z)}{z} = R_0 + \sum_{i=1}^{P} \frac{R_i}{z - p_i}
$$

where $P$ is the number of poles in $X(z)$ and

$$
R_0 = \lim_{z \to 0} X(z) \quad R_i = \text{Res}_{z=p_i} \left[ \frac{X(z)}{z} \right]
$$
The function \( X(z)/z \) can be expanded into partial fractions as

\[
\frac{X(z)}{z} = \frac{R_0}{z} + \sum_{i=1}^{P} \frac{R_i}{z - p_i}
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\[
R_0 = \lim_{z \to 0} X(z) \quad R_i = \text{Res}_{z=p_i} \left[ \frac{X(z)}{z} \right]
\]

Hence

\[
X(z) = R_0 + \sum_{i=1}^{P} \frac{R_i z}{z - p_i}
\]
Use of Partial Fractions, Technique I \textit{Cont’d}

\[ X(z) = R_0 + \sum_{i=1}^{P} \frac{R_i z}{z - p_i} \]

Therefore,

\[ x(nT) = \mathcal{Z}^{-1} \left( R_0 + \sum_{i=1}^{P} \frac{R_i z}{z - p_i} \right) = \mathcal{Z}^{-1} R_0 + \sum_{i=1}^{P} \mathcal{Z}^{-1} \frac{R_i z}{z - p_i} \]

and from the table of standard \( z \) transforms, we get

\[ x(nT) = R_0 \delta(nT) + \sum_{i=1}^{P} u(nT) R_i p_i^n \]
Example – Partial Fractions Method

Using Technique I, find the inverse $z$ transform of

$$X(z) = \frac{z}{z^2 + z + \frac{1}{2}}$$

**Solution** On expanding $X(z)/z$ into partial fractions, we get

$$\frac{X(z)}{z} = \frac{1}{z^2 + z + \frac{1}{2}} = \frac{1}{(z - p_1)(z - p_2)} = \frac{R_1}{z - p_1} + \frac{R_2}{z - p_2}$$

where

$$p_1 = \frac{e^{j3\pi/4}}{\sqrt{2}} \quad \text{and} \quad p_2 = \frac{e^{-j3\pi/4}}{\sqrt{2}}$$

Thus we obtain

$$R_1 = \text{Res}_{z=p_1} \left[ \frac{X(z)}{z} \right] = -j \quad \text{and} \quad R_2 = \text{Res}_{z=p_2} \left[ \frac{X(z)}{z} \right] = j$$

**Note:** Complex conjugate poles give complex conjugate residues.
From the table of $z$ transforms, we can now obtain

$$x(nT) = u(nT)(-jp_1^n + jp_2^n)$$

$$= \left(\frac{1}{2}\right)^{n/2} u(nT) e^{j3\pi n/4} - e^{-j3\pi n/4}$$

$$= 2 \left(\frac{1}{2}\right)^{n/2} u(nT) \sin \frac{3\pi n}{4}$$
An alternative approach is to expand $X(z)$ into partial fractions as

$$X(z) = R_0 + \sum_{i=1}^{P} \frac{R_i}{z - p_i}$$

where

$$R_0 = \lim_{z \to \infty} X(z) \quad R_i = \text{Res}_{z=p_i} X(z)$$

and $P$ is the number of poles in $X(z)$. 
An alternative approach is to expand $X(z)$ into partial fractions as

$$X(z) = R_0 + \sum_{i=1}^{P} \frac{R_i}{z - p_i}$$

where

$$R_0 = \lim_{z \to \infty} X(z) \quad R_i = \text{Res}_{z=p_i} X(z)$$

and $P$ is the number of poles in $X(z)$.

Thus

$$x(nT) = Z^{-1} \left[ R_0 + \sum_{i=1}^{P} \frac{R_i}{z - p_i} \right]$$

$$= Z^{-1} R_0 + \sum_{i=1}^{P} Z^{-1} \frac{R_i}{z - p_i}$$
\[ x(nT) = \mathcal{Z}^{-1} R_0 + \sum_{i=1}^{P} \mathcal{Z}^{-1} \frac{R_i}{z - p_i} \]

Therefore, from Table 3.2, we obtain

\[ X(nT) = R_0 \delta(nT) + \sum_{i=1}^{P} u(nT - T) R_i p_i^{n-1} \]
Example – Partial Fractions Method

Using Technique II, find the inverse $z$ transform of

$$X(z) = \frac{z}{(z - \frac{1}{2})(z - \frac{1}{4})}$$

**Solution** $X(z)$ can be expressed as

$$X(z) = \frac{z}{(z - \frac{1}{2})(z - \frac{1}{4})} = R_0 + \frac{R_1}{z - \frac{1}{2}} + \frac{R_2}{z - \frac{1}{4}}$$

where

$$R_0 = \lim_{z \to \infty} X(z) = \lim_{z \to \infty} \frac{z}{(z - \frac{1}{2})(z - \frac{1}{4})} = \lim_{z \to \infty} \frac{1}{z} = 0$$

$$R_1 = \text{Res}_{z=\frac{1}{2}} X(z) = \left. \frac{z}{(z - \frac{1}{4})} \right|_{z=\frac{1}{2}} = 2$$
Example \textit{Cont’d}

\[
R_0 = \lim_{z \to \infty} X(z) = \lim_{z \to \infty} \frac{z}{(z - \frac{1}{2})(z - \frac{1}{4})} = \lim_{z \to \infty} \frac{1}{z} = 0
\]

\[
R_1 = \text{Res}_{z = \frac{1}{2}} X(z) = \frac{z}{(z - \frac{1}{4})} \bigg|_{z = \frac{1}{2}} = 2
\]

and

\[
R_2 = \text{Res}_{z = \frac{1}{4}} X(z) = \frac{z}{(z - \frac{1}{2})} \bigg|_{z = \frac{1}{4}} = -1
\]

Hence

\[
X(z) = \frac{2}{z - \frac{1}{2}} + \frac{-1}{z - \frac{1}{4}}
\]

and from Table 3.2

\[
x(nT) = 4u(nT - T) \left[ \left( \frac{1}{2} \right)^n - \left( \frac{1}{4} \right)^n \right]
\]
The partial fraction method is based on the assumption that the denominator degree of the \( z \) transform is equal to or greater than the numerator degree.
The partial fraction method is based on the assumption that the denominator degree of the $z$ transform is equal to or greater than the numerator degree.

If this is not the case, then through long division the $z$ transform can be expressed as

$$X(z) = \frac{N(z)}{D(z)} = Q(z) + R(z)$$

where

$$Q(z) = x[(N - M) T]z^{(M-N)} + x[(N - M + 1) T]z^{(M-N-1)} + \cdots + x(0)$$

is the *quotient* polynomial and

$$R(z) = \frac{N'(z)}{D(z)}$$

is the *remainder* whose denominator degree is greater than the denominator degree.
Given a $z$ transform $X(z)$, a partial fraction expansion can be obtained through the following steps:

- represent the residues by variables,
- generate a system of simultaneous equations, and then
- solve the system of equations for the residues.
For example, if

\[ X(z) = \frac{z^2 - 2}{(z - 1)(z - 2)} \]  

(A)

we can write

\[
X(z) = R_0 + \frac{R_1}{z - 1} + \frac{R_2}{z - 2}
\]

\[
= \frac{R_0(z - 1)(z - 2) + R_1 z - 2R_1 + R_2 z - R_2}{(z - 1)(z - 2)}
\]

\[
= \frac{R_0(z^2 - 3z + 2) + R_1 z - 2R_1 + R_2 z - R_2}{(z - 1)(z - 2)}
\]

\[
= \frac{R_0 z^2 - 3R_0 z + 2R_0 + R_1 z - 2R_1 + R_2 z - R_2}{(z - 1)(z - 2)}
\]

\[
= \frac{R_0 z^2 + (R_1 + R_2 - 3R_0) z + 2R_0 - 2R_1 - R_2}{(z - 1)(z - 2)}
\]  

(B)
By equating equal powers of $z$ in Eqs. (A) and (B), we get

$$z^2 : \quad R_0 = 1$$
$$z^1 : \quad R_1 + R_2 - 3R_0 = 0$$
$$z^0 : \quad 2R_0 - 2R_1 - R_2 = -2$$
By equating equal powers of $z$ in Eqs. (A) and (B), we get

\begin{align*}
  z^2 : & \quad R_0 = 1 \\
  z^1 : & \quad R_1 + R_2 - 3R_0 = 0 \\
  z^0 : & \quad 2R_0 - 2R_1 - R_2 = -2
\end{align*}

Solving this system of equations would give the correct solution as

$$R_0 = 1, \quad R_1 = 1, \quad R_2 = 2$$
For a $z$ transform with six poles, a set of 6 simultaneous equations with 6 unknowns would need to be solved.
For a z transform with six poles, a set of 6 simultaneous equations with 6 unknowns would need to be solved.

Obviously, this is a very *inefficient method* and it should definitely be avoided.
The quick solution for this example is easily obtained by evaluating the residues individually, as follows:

\[ R_0 = \left. \frac{z^2 - 2}{(z - 1)(z - 2)} \right|_{z=\infty} = 1, \quad R_1 = \left. \frac{z^2 - 2}{(z - 2)} \right|_{z=1} = 1 \]

\[ R_2 = \left. \frac{z^2 - 2}{(z - 1)} \right|_{z=2} = 2 \]
The quick solution for this example is easily obtained by evaluating the residues individually, as follows:

\[
R_0 = \frac{z^2 - 2}{(z - 1)(z - 2)} \bigg|_{z=\infty} = 1, \quad R_1 = \frac{z^2 - 2}{(z - 2)} \bigg|_{z=1} = 1
\]

\[
R_2 = \frac{z^2 - 2}{(z - 1)} \bigg|_{z=2} = 2
\]

Hence

\[
X(z) = \frac{z^2 - 2}{(z - 1)(z - 2)} = R_0 + \frac{R_1}{z - 1} + \frac{R_2}{z - 2}
\]

\[
= 1 + \frac{1}{z - 1} + \frac{2}{z - 2}
\]
In the partial-fraction method, constant $R_0$ must always be included although it may sometimes be found to be zero.
In the partial-fraction method, constant $R_0$ must always be included although it may sometimes be found to be zero.

For example, if $R_0$ were omitted in the partial-fraction expansion

$$X(z) = \frac{z^2 - 2}{(z - 1)(z - 2)} = R_0 + \frac{R_1}{z - 1} + \frac{R_2}{z - 2}$$

then the right-hand side would assume the form

$$\frac{R_1}{z - 1} + \frac{R_2}{z - 2} = \frac{(R_1 + R_2)z - (2R_1 + R_2)}{(z - 1)(z - 2)}$$

which cannot represent the given function whatever the values of $R_1$ and $R_2$!
By the way, you can always check your work by combining the partial fractions back into a function, as you can check a division by multiplying.
This slide concludes the presentation. Thank you for your attention.