Chapter 4
THE Z TRANSFORM
4.9 Spectral Representation of Discrete-Time Signals

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- The *frequency spectrum* of a discrete-time signal is given by

$$X(z)\big|_{z=e^{j\omega T}} = X(e^{j\omega T})$$

and it is a complex quantity in general.

Spectral Representation of Discrete-Time Signals

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- The magnitude and angle of $X(e^{j\omega T})$, i.e.,

$$A(\omega) = |X(e^{j\omega T})| \quad \text{and} \quad \phi(\omega) = \arg X(e^{j\omega T})$$

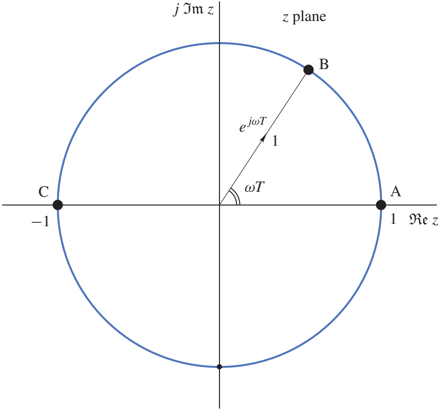
define the *amplitude spectrum* and *phase spectrum* of the discrete-time signal $x(nT)$, respectively.

- The exponential function $e^{j\omega T}$ is a complex number of magnitude 1 and angle ωT and as ω is increased from zero to $2\pi/T$, $e^{j\omega T}$ will trace a circle of radius 1 in the z plane, which is referred to as the *unit circle*.

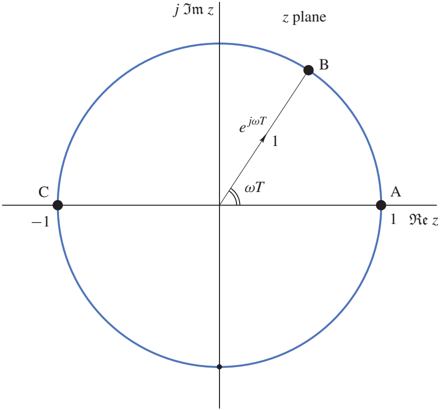
Spectral Representation *Cont'd*

- The exponential function $e^{j\omega T}$ is a complex number of magnitude 1 and angle ωT and as ω is increased from zero to $2\pi/T$, $e^{j\omega T}$ will trace a circle of radius 1 in the z plane, which is referred to as the *unit circle*.
- In effect, the frequency spectrum of a discrete-time signal, $x(nT)$, can be deduced by evaluating its z transform, $X(z)$, on the unit circle.

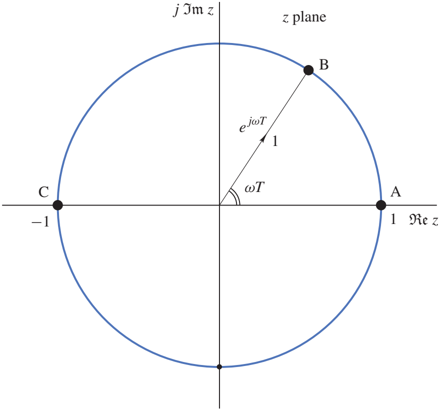
- If $\omega = 0$, then $e^{j\omega T} = e^0 = 1$, i.e., point A corresponds to zero frequency.



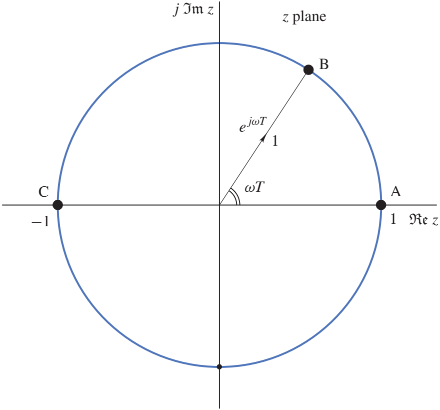
- At half the sampling frequency, $\omega = \omega_s/2 = \pi/T$ and hence $e^{j\omega T} = e^{j\pi} = -1$, i.e., point C corresponds to the Nyquist frequency.



- At the sampling frequency, $\omega = \omega_s = 2\pi/T$ and hence $e^{j\omega T} = e^{j2\pi} = 1$, i.e., point A also corresponds to the sampling frequency.



- If vector $e^{j\omega T}$ is rotated k complete revolution starting from some arbitrary point B, it will return to point B.



Hence

$$e^{j(\omega T+2\pi k)} = e^{(j\omega+2\pi k/T)T} = e^{(j\omega+k\omega_s)T} = e^{j\omega T}$$

and, therefore,

$$X(e^{j(\omega+k\omega_s)T}) = X(e^{j\omega T})$$

In effect, *the frequency spectrum of a discrete-time signal is a periodic function of frequency with period ω_s*.

Example

Obtain expressions for the frequency, amplitude, and phase spectrums of the signal

$$x(nT) = u(nT)e^{-\alpha nT} \sin \omega_0 nT$$

Solution The z transform of the signal can be obtained from Table 3.2 as

$$X(z) = \frac{ze^{-\alpha T} \sin \omega_0 T}{z^2 - 2ze^{-\alpha T} \cos \omega_0 T + e^{-2\alpha T}}$$

$X(z)$ can be expressed as

$$X(z) = \frac{a_1 z}{z^2 + b_1 z + b_0}$$

where

$$a_1 = e^{-\alpha T} \sin \omega_0 T \quad b_0 = e^{-2\alpha T} \quad b_1 = -2e^{-\alpha T} \cos \omega_0 T$$

...

$$X(z) = \frac{a_1 z}{z^2 + b_1 z + b_0}$$

Hence the frequency spectrum of the signal is given by

$$\begin{aligned} X(e^{j\omega T}) &= \frac{a_1 e^{j\omega T}}{e^{j2\omega T} + b_1 e^{j\omega T} + b_0} \\ &= \frac{a_1 e^{j\omega T}}{\cos 2\omega T + j \sin 2\omega T + b_1 \cos \omega T + j b_1 \sin \omega T + b_0} \\ &= \frac{a_1 e^{j\omega T}}{b_0 + b_1 \cos \omega T + \cos 2\omega T + j(b_1 \sin \omega T + \sin 2\omega T)} \quad \blacksquare \end{aligned}$$

The amplitude and phase spectrums can be deduced by letting

$$X(e^{j\omega T}) = \frac{a_1 e^{j\omega T}}{b_0 + b_1 \cos \omega T + \cos 2\omega T + j(b_1 \sin \omega T + \sin 2\omega T)}$$
$$= A(\omega) e^{j\phi(\omega)}$$

Hence

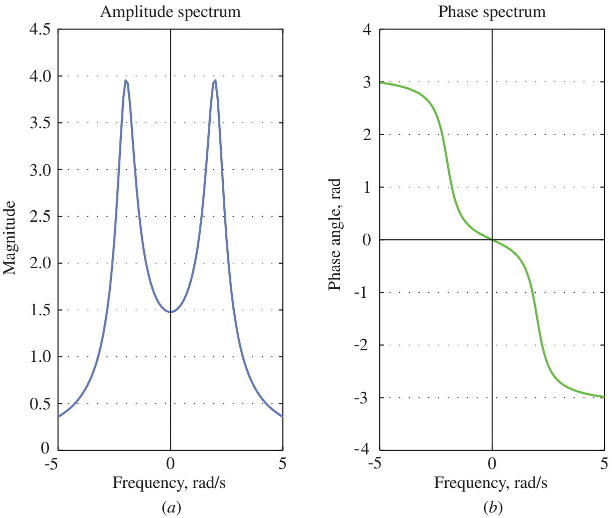
$$A(\omega) = \frac{|a_1| \cdot |e^{j\omega T}|}{|(b_0 + b_1 \cos \omega T + \cos 2\omega T) + j(b_1 \sin \omega T + \sin 2\omega T)|}$$
$$= \frac{|a_1|}{\sqrt{(b_0 + b_1 \cos \omega T + \cos 2\omega T)^2 + (b_1 \sin \omega T + \sin 2\omega T)^2}}$$
$$= \frac{|a_1|}{\sqrt{1 + b_0^2 + b_1^2 + 2b_1(1 + b_0) \cos \omega T + 2b_0 \cos 2\omega T}} \quad \blacksquare$$

Example *Cont'd*

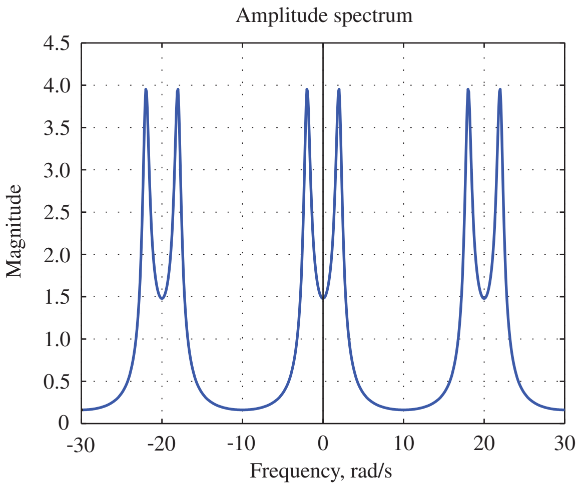
$$\begin{aligned}\phi(\omega) &= \arg(a_1) + \arg e^{j\omega T} - \arg[b_0 + b_1 \cos\omega T + \cos 2\omega T \\ &\quad + j(b_1 \sin\omega T + \sin 2\omega T)] \\ &= \arg a_1 + \omega T - \tan^{-1}\frac{b_1 \sin\omega T + \sin 2\omega T}{b_0 + b_1 \cos\omega T + \cos 2\omega T} \quad \blacksquare\end{aligned}$$

where

$$\arg a_1 = \begin{cases} 0 & \text{if } a_1 \geq 0 \\ -\pi & \text{otherwise} \end{cases}$$



Amplitude and phase spectrums.



(*a*)

Periodicity of amplitude spectrum.

This slide concludes the presentation.
Thank you for your attention.