Chapter 3
THE Z TRANSFORM
3.1 Introduction 3.2 Definition 3.3 Convergence Properties 3.4 The Z Transform as a Laurent Series 3.5 Inverse Z Transform 3.6 Theorems and Properties 3.7 Elementary Discrete-Time Signals

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Introduction

- The Fourier series and Fourier transform can be used to obtain spectral representations for periodic and nonperiodic *continuous-time signals*, respectively (see Chap. 2).

  Analogous spectral representations can be obtained for discrete-time signals by using the z transform.

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- The Fourier series and Fourier transform can be used to obtain spectral representations for periodic and nonperiodic *continuous-time signals*, respectively (see Chap. 2).

  Analogous spectral representations can be obtained for discrete-time signals by using the z transform.

- The Fourier transform will convert a real continuous-time signal into a function of complex variable $j\omega$.

  Similarly, the z transform will convert a real *discrete-time signal* into a function of complex variable z.

- The z transform, like the Fourier transform, comes along with an inverse transform, namely, the inverse z transform.

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  Consequently, a discrete-time signal can be readily recovered from its z transform.

- The availability of an inverse makes the z transform very useful for the representation of digital filters and discrete-time systems in general.

Introduction *Cont'd*

- The most basic representation of discrete-time systems is in terms of difference equations (see Chap. 4) but through the use of the z transform, difference equations can be reduced to algebraic equations which are much easier to handle.

Objectives

- Definition of Z Transform
- Convergence Properties
- The Z Transform as a Laurent series
- Inverse Z Transform
- Theorems and Properties
- Elementary Functions
- Examples

- Consider a bounded discrete-time signal $x(nT)$ that satisfies the conditions

$$\begin{array}{ll}
\text{(i)} & x(nT) = 0 \quad \text{for} \quad n < -N_1 \\
\text{(ii)} & |x(nT)| \leq K_1 \quad \text{for} \quad -N_1 \leq n < N_2 \\
\text{(iii)} & |x(nT)| \leq K_2 r^n \quad \text{for} \quad n \geq N_2
\end{array}$$

where N_1 and N_2 are positive integers and r is a positive constant.

...

$$
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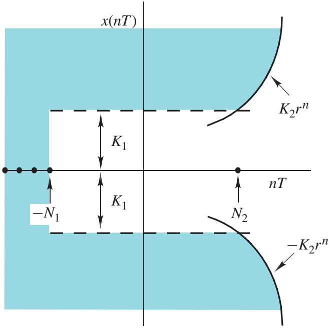

(a)

- The z transform of a discrete-time signal $x(nT)$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(nT) z^{-n}$$

for all z for which $X(z)$ converges.

- Although the z transform of a signal $x(nT)$ is an infinite series, in practice it can be represented in terms of a rational function as

$$X(z) = \sum_{n=-\infty}^{\infty} x(nT)z^{-n}$$
$$= \frac{N(z)}{D(z)} = \frac{\sum_{i=0}^{M} a_i z^{M-i}}{z^N + \sum_{i=1}^{N} b_i z^{N-i}} = H_0 \frac{\prod_{i=1}^{M}(z - z_i)}{\prod_{i=1}^{N}(z - p_i)}$$

where z_i and p_i are the zeros and poles of the z transform and H_0 is a multiplier constant.

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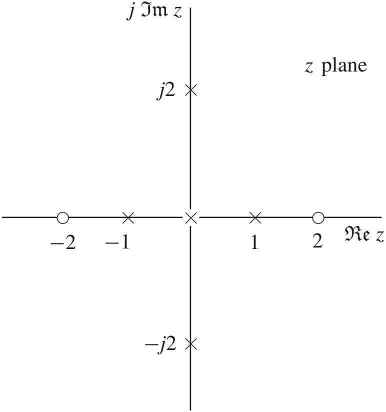
where z_i and p_i are the zeros and poles of the z transform and H_0 is a multiplier constant.

- In effect, z transforms can be represented by *zero-pole plots*.

Example

The following z transform has the zero-pole plot shown.

$$X(z) = \frac{(z^2-4)}{z(z^2-1)(z^2+4)} = \frac{(z-2)(z+2)}{z(z-1)(z+1)(z-j2)(z+j2)}$$



Theorem 3.1 Absolute Convergence

If

$$\begin{array}{ll} \text{(i)} & x(nT) = 0 \quad \text{for} \quad n < -N_1 \\ \text{(ii)} & |x(nT)| \le K_1 \quad \text{for} \quad -N_1 \le n < N_2 \\ \text{(iii)} & |x(nT)| \le K_2 r^n \quad \text{for} \quad n \ge N_2 \end{array}$$

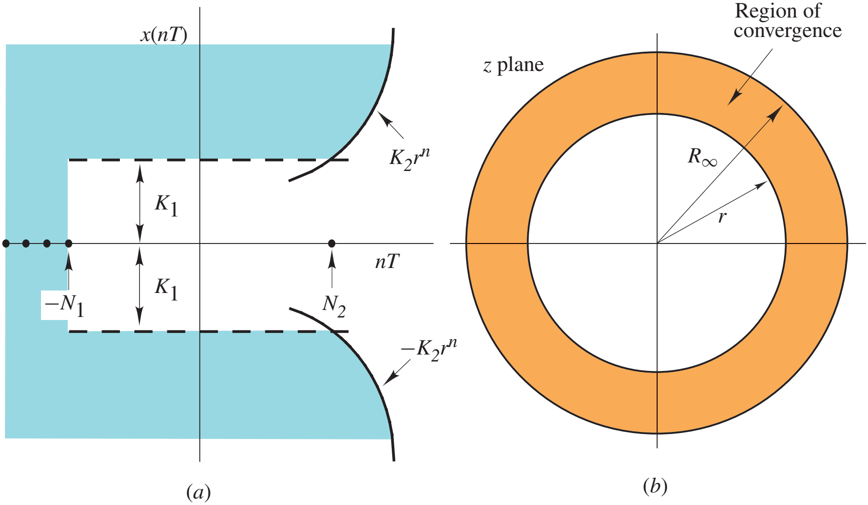
where N_1 and N_2 are positive constants and r is the smallest positive constant that will satisfy condition (iii), then the z transform of $x(nT)$, i.e.,

$$X(z) = \sum_{n=-\infty}^{\infty} x(nT) z^{-n}$$

exists and converges absolutely if and only if

$$r < |z| < R_\infty \quad \text{with} \quad R_\infty \to \infty$$

Absolute Convergence *Cont'd*



Absolute Convergence *Cont'd*

The proofs of the Absolute Convergence Theorem and the theorems that follow can be found in the textbook.

The Z Transform as a Laurent Series

- The Laurent series of a function $X(z)$ about point $z = a$ assumes the form

$$X(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^{-n}$$

(See Appendix A.)

The Z Transform as a Laurent Series

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$$X(z) = \sum_{n=-\infty}^{\infty} a_n (z - a)^{-n}$$

(See Appendix A.)

- The z transform is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(nT) z^{-n}$$

If we compare the above two series for $X(z)$, we conclude that the z transform is a Laurent series of $X(z)$ about the origin, i.e., $a = 0$, with

$$a_n = x(nT)$$

The Z Transform as a Laurent Series *Cont'd*

- Since the z transform is a specific Laurent series, it follows that *it inherits all the properties* of the Laurent series, which are stated in the Laurent theorem as detailed in the slides that follow.

Laurent Theorem

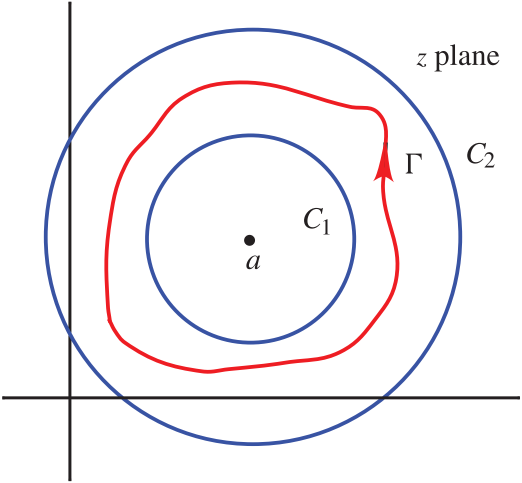
(a) If $F(z)$ is an analytic and single-valued function on two concentric circles C_1 and C_2 with center a and in the annulus between them, then it can be represented by the Laurent series

$$F(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^{-n}$$

where

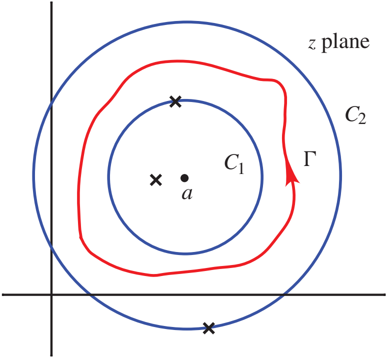
$$a_n = \frac{1}{2\pi j} \oint_{\Gamma} F(z)(z-a)^{n-1}\, dz$$

The contour of integration Γ is a closed contour in the counterclockwise sense lying in the annulus between circles C_1 and C_2 and encircling the inner circle.



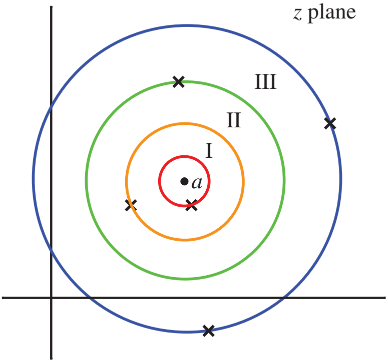
(a)

(b) The Laurent series converges and represents $F(z)$ in the open annulus obtained by continuously increasing the radius of C_2 and decreasing the radius of C_1 until each of C_1 and C_2 reaches a point where $F(z)$ is singular.



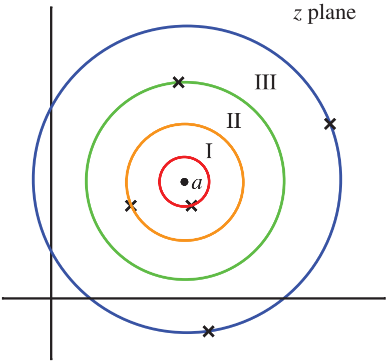
(*b*)

(c) A function $F(z)$ can have several, possibly many, annuli of convergence about a given point $z = a$ and for each one a Laurent series can be obtained.



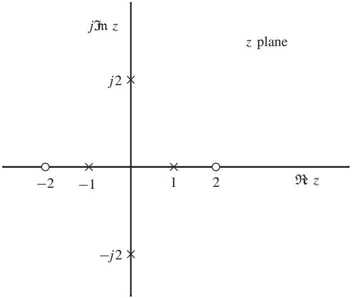
(c)

(d) The Laurent series for a given annulus of convergence is unique.

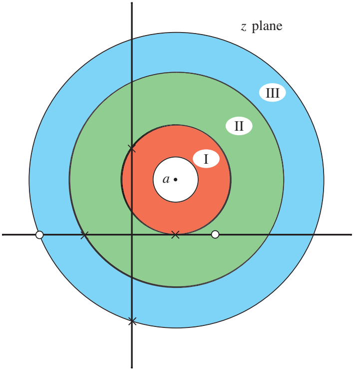


(c)

The function represented by the zero-pole plot at the left has three unique Laurent series as shown at the right.



(a)



Inverse Z Transform

- The absolute-convergence theorem states that the z transform, $X(z)$, of a discrete-time signal $x(nT)$ satisfying the conditions

$$\begin{array}{ll}
\text{(i)} & x(nT) = 0 \quad \text{for} \quad n < -N_1 \\
\text{(ii)} & |x(nT)| \leq K_1 \quad \text{for} \quad -N_1 \leq n < N_2 \\
\text{(iii)} & |x(nT)| \leq K_2 r^n \quad \text{for} \quad n \geq N_2
\end{array}$$

  exists and converges absolutely if and only if

$$r < |z| < R \quad \text{with} \quad R \to \infty$$

- The Laurent theorem states that a function $X(z)$ has as many distinct Laurent series about the origin as there are annuli of convergence.

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- One of these series converges in the outer annulus (i.e., the largest one) which is defined as

$$R_0 < |z| < R \quad \text{with} \quad R \to \infty$$

where R_0 is the radius of a circle passing through the most distant pole of $X(z)$ from the origin.

Summarizing:

- From the absolute convergence theorem, the z transform converges in the annulus

$$r < |z| < R \quad \text{with} \quad R \to \infty$$

Inverse Z Transform *Cont'd*

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- From the absolute convergence theorem, the z transform converges in the annulus

$$r < |z| < R \quad \text{with} \quad R \to \infty$$

- From the Laurent theorem, there is a unique Laurent series of $X(z)$ that converges in the outer annulus of convergence

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Inverse Z Transform *Cont'd*

Summarizing:

- From the absolute convergence theorem, the z transform converges in the annulus

$$r < |z| < R \quad \text{with} \quad R \to \infty$$

- From the Laurent theorem, there is a unique Laurent series of $X(z)$ that converges in the outer annulus of convergence

$$R_0 < |z| < R \quad \text{with} \quad R \to \infty$$

- Therefore, *the z transform of $x(nT)$ is the unique Laurent series that converges in the outer annulus* and, furthermore, $r = R_0$.

Inverse Z Transform *Cont'd*

- We conclude that signal $x(nT)$ can be obtained from its z transform $X(z)$ by finding the coefficients of the Laurent series of $X(z)$ that converges in the outer annulus.

Inverse Z Transform *Cont'd*

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- From the Laurent theorem, we have

$$x(nT) = \frac{1}{2\pi j} \oint_{\Gamma} X(z) z^{n-1}\, dz$$

where contour Γ encloses all the poles of $X(z)z^{n-1}$.

Inverse Z Transform *Cont'd*

- We conclude that signal $x(nT)$ can be obtained from its z transform $X(z)$ by finding the coefficients of the Laurent series of $X(z)$ that converges in the outer annulus.
- From the Laurent theorem, we have

$$x(nT) = \frac{1}{2\pi j} \oint_{\Gamma} X(z) z^{n-1}\, dz$$

  where contour Γ encloses all the poles of $X(z)z^{n-1}$.
- In DSP, this contour integral is said to be the *inverse z transform* of $X(z)$.

Notation

- Like the Fourier transform and its inverse, the z transform and its inverse are often represented in terms of operator notation as
$$X(z) = \mathcal{Z}x(nT) \quad \text{and} \quad x(nT) = \mathcal{Z}^{-1}X(z)$$
respectively.

- The general properties of the z transform can be described in terms of a small number of theorems, as detailed in the slides that follow.

Z Transform Theorems

- The general properties of the z transform can be described in terms of a small number of theorems, as detailed in the slides that follow.
- In these theorems

$$\mathcal{Z}x(nT) = X(z) \quad \mathcal{Z}x_1(nT) = X_1(z) \quad \mathcal{Z}x_2(nT) = X_2(z)$$

and a, b, w, and K represent constants which may be complex.

Theorem 3.3 Linearity

- The z transform of a linear combination of discrete-time signals is given by

$$\mathcal{Z}[ax_1(nT) + bx_2(nT)] = aX_1(z) + bX_2(z)$$

Theorem 3.3 Linearity

- The z transform of a linear combination of discrete-time signals is given by

$$\mathcal{Z}[ax_1(nT) + bx_2(nT)] = aX_1(z) + bX_2(z)$$

- Similarly, the inverse z transform of a linear combination of z transforms is given by

$$\mathcal{Z}^{-1}[aX_1(z) + bX_2(z)] = ax_1(nT) + bx_2(nT)$$

Theorem 3.4 Time Shifting

- For any positive or negative integer m,

$$\mathcal{Z}x(nT + mT) = z^m X(z)$$

In effect, multiplying the z transform of a signal by a negative or positive power of z will cause the signal to be *delayed or advanced* by mT s.

Theorem 3.5 Complex Scale Change

- For an arbitrary real or complex constant w

$$\mathcal{Z}[w^{-n}x(nT)] = X(wz)$$

Evidently, multiplying a discrete-time signal by w^{-n} *is equivalent to replacing z by wz* in its z transform.

Similarly, multiplying a discrete-time signal by v^n is *equivalent to replacing z by z/v* in its z transform.

Theorem 3.6 Complex Differentiation

- The z transform of an arbitrary signal $nT_1x(nT)$ is given by

$$\mathcal{Z}[nT_1x(nT)] = -T_1z\frac{dX(z)}{dz}$$

  Complex differentiation provides a simple way of obtaining the z transform of a discrete-time signal that can be expressed as a product $nT_1x(nT)$.

Theorem 3.7 Real Convolution

- The z transform of the real convolution summation of two signals $x_1(kT)$ and $x_2(nT)$ is given by

$$\mathcal{Z}\sum_{k=-\infty}^{\infty} x_1(kT)x_2(nT-kT) = \mathcal{Z}\sum_{k=-\infty}^{\infty} x_1(nT-kT)x_2(kT)$$
$$= X_1(z)X_2(z)$$

  The real convolution summation is used frequently for the representation of digital filters and discrete-time systems (see Chap. 4).

Theorem 3.8 Initial-Value Theorem

- The initial value of a signal $x(nT)$ represented by a z transform of the form

$$X(z) = \frac{N(z)}{D(z)} = \frac{\sum_{i=0}^{M} a_i z^{M-i}}{\sum_{i=0}^{N} b_i z^{N-i}}$$

occurs at

$$KT = (N - M)T$$

and its value at $nT = KT$ is given by

$$x(KT) = \lim_{z \to \infty} [z^K X(z)]$$

...

$$X(z) = \frac{N(z)}{D(z)} = \frac{\sum_{i=0}^{M} a_i z^{M-i}}{\sum_{i=0}^{N} b_i z^{N-i}}$$

- Corollary: If the degree of the numerator polynomial, $N(z)$, in a z transform is equal to or less than the degree of the denominator polynomial $D(z)$, then we have

$$x(nT) = 0 \quad \text{for} \quad n < 0$$

i.e., the signal is right-sided.

Theorem 3.9 Final-Value Theorem

- The value of $x(nT)$ as $n \to \infty$ is given by

$$x(\infty) = \lim_{z \to 1} [(z-1)X(z)]$$

The final-value theorem can be used to determine the steady-state response of a discrete-time system.

Theorem 3.10 Complex Convolution

- If the z transforms of two discrete-time signals $x_1(nT)$ and $x_2(nT)$ are available, then the z transform of their product, $X_3(z)$, can be obtained as

$$\begin{aligned} X_3(z) = \mathcal{Z}[x_1(nT)x_2(nT)] &= \frac{1}{2\pi j} \oint_{\Gamma_1} X_1(v) X_2\left(\frac{z}{v}\right) v^{-1}\, dv \\ &= \frac{1}{2\pi j} \oint_{\Gamma_2} X_1\left(\frac{z}{v}\right) X_2(v) v^{-1}\, dv \end{aligned}$$

where Γ_1 $(or \Gamma_2)$ is a contour in the common region of convergence of $X_1(v)$ and $X_2(z/v)$ (or $X_1(z/v)$ and $X_2(v)$).

- The complex convolution theorem can be used to obtain the z transform of a product of discrete-time signals whose z transforms are available.

Theorem 3.10 Complex Convolution *Cont'd*

- The complex convolution theorem can be used to obtain the z transform of a product of discrete-time signals whose z transforms are available.
- It is also the basis of the window method for the design of nonrecursive digital filters (see Chap. 9).

Theorem 3.11 Parseval's Discrete-Time Formula

- If $X(z)$ is the z transform of a discrete-time signal $x(nT)$, then

$$\sum_{n=-\infty}^{\infty} |x(nT)|^2 = \frac{1}{\omega_s} \int_0^{\omega_s} |X(e^{j\omega T})|^2 \, d\omega$$

where $\omega_s = 2\pi/T$.

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where $\omega_s = 2\pi/T$.

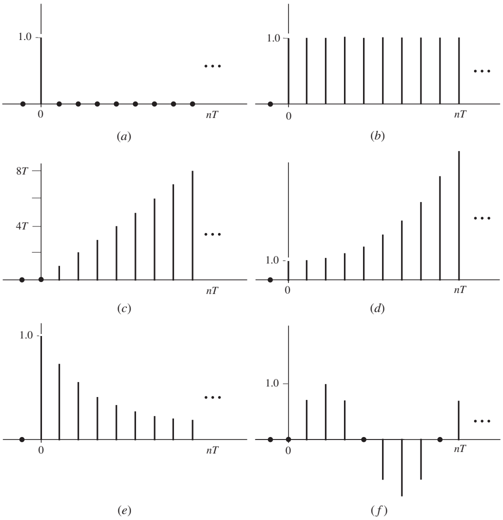
- Parseval's formula is often used to solve a problem known as *scaling* which is associated with the design of recursive digital filters in hardware form (see Chap. 14).

- If T is normalized to 1 s, Parseval's formula simplifies to:

$$\sum_{n=-\infty}^{\infty} |x(nT)|^2 = \frac{1}{2\pi}\int_0^{2\pi} |X(e^{j\omega T})|^2 \, d\omega$$

Elementary Discrete-Time Signals

Function	Definition
Unit impulse	$\delta(nT) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$
Unit step	$u(nT) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$
Unit ramp	$r(nT) = \begin{cases} nT & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$
Exponential	$u(nT)e^{\alpha nT}, \ (\alpha > 0)$
Exponential	$u(nT)e^{\alpha nT}, \ (\alpha < 0)$
Sinusoid	$u(nT)\sin \omega nT$



(*a*) Unit impulse, (*b*) unit step, (*c*) unit ramp, (*d*) increasing exponential (*e*) decreasing exponential, (*c*) sinusoid.

Examples

Find the z transforms of the following signals:

(a) unit-impulse $\delta(nT)$

(b) unit-step $u(nT)$

(c) delayed unit-step $u(nT - kT)K$

(d) signal $u(nT)Kw^n$

(e) exponential signal $u(nT)e^{-\alpha nT}$

(f) unit-ramp $r(nT)$

(g) sinusoidal signal $u(nT)\sin \omega nT$

Solutions

(a) From the definitions of the z transform and $\delta(nT)$, we have

$$\mathcal{Z}\delta(nT) = \delta(0) + \delta(T)z^{-1} + \delta(2T)z^{-2} + \cdots = 1 \quad \blacksquare$$

Solutions

(*a*) From the definitions of the z transform and $\delta(nT)$, we have

$$\mathcal{Z}\delta(nT) = \delta(0) + \delta(T)z^{-1} + \delta(2T)z^{-2} + \cdots = 1 \quad \blacksquare$$

(*b*) As in part (*a*)

$$\begin{aligned}\mathcal{Z}u(nT) &= u(0) + u(T)z^{-1} + u(2T)z^{-2} + \cdots \\ &= 1 + z^{-1} + z^{-2} + \cdots = (1 - z^{-1})^{-1} \\ &= \frac{z}{z-1} \quad \blacksquare\end{aligned}$$

Solutions

(*a*) From the definitions of the z transform and $\delta(nT)$, we have

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$$\begin{aligned} \mathcal{Z}u(nT) &= u(0) + u(T)z^{-1} + u(2T)z^{-2} + \cdots \\ &= 1 + z^{-1} + z^{-2} + \cdots = (1 - z^{-1})^{-1} \\ &= \frac{z}{z-1} \quad \blacksquare \end{aligned}$$

(*c*) From the time-shifting theorem (Theorem 3.4) and part (*b*), we have

$$\mathcal{Z}[u(nT - kT)K] = Kz^{-k}\mathcal{Z}u(nT) = \frac{Kz^{-(k-1)}}{z-1} \quad \blacksquare$$

(*d*) From the complex-scale-change theorem (Theorem 3.5) and part (*b*), we get

$$\mathcal{Z}[u(nT)Kw^n] = K\mathcal{Z}\left[\left(\frac{1}{w}\right)^{-n} u(nT)\right]$$

$$= K\mathcal{Z}u(nT)\big|_{z\to z/w} = \frac{Kz}{z-w} \quad \blacksquare$$

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$$= K\mathcal{Z}u(nT)\big|_{z\to z/w} = \frac{Kz}{z-w} \quad \blacksquare$$

(*e*) By letting $K = 1$ and $w = e^{-\alpha T}$ in part (*d*), we obtain

$$\mathcal{Z}[u(nT)e^{-\alpha nT}] = \frac{z}{z-e^{-\alpha T}} \quad \blacksquare$$

(*f*) From the complex-differentiation theorem (Theorem 3.6) and part (*b*), we have

$$\mathcal{Z}r(nT) = \mathcal{Z}[nTu(nT)] = -Tz\frac{d}{dz}[\mathcal{Z}u(nT)]$$

$$= -Tz\frac{d}{dz}\left[\frac{z}{(z-1)}\right] = \frac{Tz}{(z-1)^2} \quad \blacksquare$$

(g) From part (e), we deduce

$$
\begin{aligned}
\mathcal{Z}[u(nT)\sin\omega nT] &= \mathcal{Z}\left[\frac{u(nT)}{2j}\left(e^{j\omega nT} - e^{-j\omega nT}\right)\right] \\
&= \frac{1}{2j}\mathcal{Z}[u(nT)e^{j\omega nT}] - \frac{1}{2j}\mathcal{Z}\left[u(nT)e^{-j\omega nT}\right] \\
&= \frac{1}{2j}\left(\frac{z}{z - e^{j\omega T}} - \frac{z}{z - e^{-j\omega T}}\right) \\
&= \frac{z\sin\omega T}{z^2 - 2z\cos\omega T + 1} \quad \blacksquare
\end{aligned}
$$

Standard Z Transforms

$x(nT)$	$X(z)$
$\delta(nT)$	1
$u(nT)$	$\dfrac{z}{z-1}$
$u(nT-kT)K$	$\dfrac{Kz^{-(k-1)}}{z-1}$
$u(nT)Kw^n$	$\dfrac{Kz}{z-w}$
$u(nT-kT)Kw^{n-1}$	$\dfrac{K(z/w)^{-(k-1)}}{z-w}$
$u(nT)e^{-\alpha nT}$	$\dfrac{z}{z-e^{-\alpha T}}$
$r(nT)$	$\dfrac{Tz}{(z-1)^2}$

Standard Z Transforms *Cont'd*

$x(nT)$	$X(z)$
$r(nT)e^{-\alpha nT}$	$\dfrac{Te^{-\alpha T}z}{(z-e^{-\alpha T})^2}$
$u(nT)\sin\omega nT$	$\dfrac{z\sin\omega T}{z^2-2z\cos\omega T+1}$
$u(nT)\cos\omega nT$	$\dfrac{z(z-\cos\omega T)}{z^2-2z\cos\omega T+1}$
$u(nT)e^{-\alpha nT}\sin\omega nT$	$\dfrac{ze^{-\alpha T}\sin\omega T}{z^2-2ze^{-\alpha T}\cos\omega T+e^{-2\alpha T}}$
$u(nT)e^{-\alpha nT}\cos\omega nT$	$\dfrac{z(z-e^{-\alpha T}\cos\omega T)}{z^2-2ze^{-\alpha T}\cos\omega T+e^{-2\alpha T}}$

This slide concludes the presentation.

Thank you for your attention.