Chapter 3
THE Z TRANSFORM
3.8 Z-Transform Inversion Techniques

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Z-Transform Inversion Techniques

- The most fundamental method for the inversion of a z transform is the *general inversion method* which is based on the Laurent theorem.
- In this method, the inverse of a z transform $X(z)$ is given by

$$x(nT) = \frac{1}{2\pi j} \oint_{\Gamma} X(z) z^{n-1} \, dz$$

where Γ is a closed contour in the counterclockwise sense enclosing all the singularities of function $X(z)z^{n-1}$.

...

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- At first sight, the above contour integration may appear to be a formidable task.
- However, for most DSP applications, the z transform turns out to be a rational function and for such functions the contour integral can be easily evaluated by using the residue theorem.

- According to the residue theorem,

$$x(nT) = \frac{1}{2\pi j} \oint_{\Gamma} X(z) z^{n-1}\, dz = \sum_{i=1}^{P} \text{res}_{\, z \to p_i} \left[X(z) z^{n-1} \right]$$

where $\text{res}_{\, z \to p_i} \left[X(z) z^{n-1} \right]$ and P are the residue of pole p_i and the number of poles of $X(z)z^{n-1}$, respectively.

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where $\text{res}_{\, z \to p_i} \left[X(z) z^{n-1} \right]$ and P are the residue of pole p_i and the number of poles of $X(z)z^{n-1}$, respectively.

- For a pole of order m_i,

$$\text{res}_{\, z = p_i} \left[X(z) z^{n-1} \right] = \frac{1}{(m_i - 1)!} \lim_{z \to p_i} \frac{d^{m_i - 1}}{dz^{m_i - 1}} \left[(z - p_i)^{m_i} X(z) z^{n-1} \right]$$

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- For a simple pole,

$$\text{res}_{\,z=p_i} \left[X(z) z^{n-1} \right] = \lim_{z\to p_i} \left[(z - p_i) X(z) z^{n-1} \right]$$

Example – General Inversion Method

Using the general inversion method, find the inverse z transform of

$$X(z) = \frac{1}{2(z-1)\left(z+\frac{1}{2}\right)}$$

Solution We note that the factor z^{n-1} introduces a pole in $X(z)z^{n-1}$ at the origin for the case $n = 0$, which must be taken into account in the evaluation of $x(0)$.

Note: For $n > 0$, the pole at the origin *disappears*.

Thus for $n = 0$, we have

$$X(z)z^{n-1}\big|_{n=0} = \left.\frac{z^{n-1}}{2(z-1)\left(z+\frac{1}{2}\right)}\right|_{n=0} = \frac{1}{2z(z-1)\left(z+\frac{1}{2}\right)}$$

Hence

$$x(0) = \left.\frac{1}{2(z-1)\left(z+\frac{1}{2}\right)}\right|_{z=0} + \left.\frac{1}{2z\left(z+\frac{1}{2}\right)}\right|_{z=1} + \left.\frac{1}{2z(z-1)}\right|_{z=-\frac{1}{2}} = -1 + \tfrac{1}{3} + \tfrac{2}{3} = 0$$

Actually, this follows from the initial-value theorem (Theorem 3.8) without any calculations.

Example *Cont'd*

For $n > 0$

$$\begin{aligned} x(nT) &= \left.\frac{z^{n-1}}{2(z+\frac{1}{2})}\right|_{z=1} + \left.\frac{z^{n-1}}{2(z-1)}\right|_{z=-\frac{1}{2}} \\ &= \tfrac{1}{3} - \tfrac{1}{3}\left(-\tfrac{1}{2}\right)^{n-1} \end{aligned}$$

and from the initial-value theorem, $x(nT) = 0$ for $n < 0$.

Therefore, for any value of n, we have

$$x(nT) = u(nT - T)\left[\tfrac{1}{3} - \tfrac{1}{3}\left(-\tfrac{1}{2}\right)^{n-1}\right] \quad \blacksquare$$

Using the general inversion method, find the inverse z transform of

$$X(z) = \frac{(2z-1)z}{2(z-1)\left(z+\frac{1}{2}\right)}$$

Solution We can write

$$X(z)z^{n-1} = \frac{(2z-1)z \cdot z^{n-1}}{2(z-1)\left(z+\frac{1}{2}\right)} = \frac{(2z-1)z^n}{2(z-1)\left(z+\frac{1}{2}\right)}$$

We note that $X(z)z^{n-1}$ has simple poles at $z = 1$ and $-\frac{1}{2}$.

Furthermore, the zero in $X(z)$ at the origin cancels the pole at the origin introduced by z^{n-1} for the case $n = 0$.

...

$$X(z)z^{n-1} = \frac{(2z-1)z^n}{2(z-1)\left(z+\frac{1}{2}\right)}$$

Hence for any $n \geq 0$, the general inversion formula gives

$$\begin{aligned} x(nT) &= \operatorname{res}_{z=1}\left[X(z)z^{n-1}\right] + \operatorname{res}_{z=-\frac{1}{2}}\left[X(z)z^{n-1}\right] \\ &= \left.\frac{(2z-1)z^n}{2\left(z+\frac{1}{2}\right)}\right|_{z=1} + \left.\frac{(2z-1)z^n}{2(z-1)}\right|_{z=-\frac{1}{2}} \\ &= \frac{1}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^n \end{aligned}$$

Example *Cont'd*

Since the numerator degree in $X(z)$ does not exceed the denominator degree, it follows that $x(nT)$ is a right-sided signal, i.e., $x(nT) = 0$ for $n < 0$, according to the Corollary of Theorem 3.8.

Therefore, for any value of n, we have

$$x(nT) = u(nT)\left[\tfrac{1}{3} + \tfrac{2}{3}\left(-\tfrac{1}{2}\right)^n\right] \quad \blacksquare$$

where $u(nT)$ is the unit-step function.

Since

- the z transform is a particular type of Laurent series, and
- the Laurent series in a given annulus of convergence is unique

it follows that *any technique* that can be used to generate a power series for $X(z)$ that converges in the outermost annulus of convergence can be used to obtain the inverse z transform.

Consequently, several inversion techniques are available, as follows:

- using the binomial theorem,
- using the convolution theorem,
- performing long division,
- using the initial-value theorem, or
- expanding $X(z)$ into partial fractions.

Some of these techniques are illustrated by examples in the next few slides.

Example – Binomial Theorem

Using the binomial method, find the inverse z transform of

$$X(z) = \frac{Kz^m}{(z-w)^k}$$

where m and k are integers, and K and w are constants, possibly complex.

Solution The inverse z transform can be obtained by obtaining a binomial series for $X(z)$ that converges in the outside annulus of $X(z)$.

Example *Cont'd*

Such a binomial series can be obtained by expressing $X(z)$ as

$$\begin{aligned} X(z) &= Kz^{m-k}[1+(-wz^{-1})]^{-k} \\ &= Kz^{m-k}\left[\binom{-k}{0}+\binom{-k}{1}(-wz^{-1})+\binom{-k}{2}(-wz^{-1})^2\right. \\ &\quad \left. +\cdots+\binom{-k}{n}(-wz^{-1})^n+\cdots\right] \end{aligned}$$

where $$\binom{-k}{n}=\frac{(-k)(-k-1)\ldots(-k-n+1)}{n!}$$

Hence

$$X(z)=\sum_{n=-\infty}^{\infty}Ku(nT)\frac{(-k)(-k-1)\cdots(-k-n+1)(-w)^n z^{-n+m-k}}{n!}$$

. . .

$$X(z) = \sum_{n=-\infty}^{\infty} Ku(nT)\frac{(-k)(-k-1)\cdots(-k-n+1)(-w)^n z^{-n+m-k}}{n!}$$

Now if we let $n = n' + m - k$ and then replace n' by n, we have

$$X(z) = \sum_{n=-\infty}^{\infty} \Bigg\{ Ku[(n+m-k)T] \times \frac{(-k)(-k-1)\cdots(-n-m+1)(-w)^{n+m-k}}{(n+m-k)!} \Bigg\} z^{-n}$$

...

$$X(z) = \sum_{n=-\infty}^{\infty} \left\{ Ku[(n+m-k)T] \right.$$
$$\left. \times \frac{(-k)(-k-1)\cdots(-n-m+1)(-w)^{n+m-k}}{(n+m-k)!} \right\} z^{-n}$$

Hence the coefficient of z^{-n} is obtained as

$$x(nT) = \mathcal{Z}^{-1}\left[\frac{Kz^m}{(z-w)^k}\right]$$
$$= Ku[(n+m-k)T]\frac{(-k)(-k-1)\cdots(-n-m+1)(-w)^{n+m-k}}{(n+m-k)!} \quad \blacksquare$$

By assigning different values to constants k, K, and m a *variety* of z-transform pairs can be deduced as shown in the next slide.

Example *Cont'd*

x(nT)	X(z)
$u(nT)$	$\dfrac{z}{z-1}$
$u(nT-kT)K$	$\dfrac{Kz^{-(k-1)}}{z-1}$
$u(nT)Kw^n$	$\dfrac{Kz}{z-w}$
$u(nT-kT)Kw^{n-1}$	$\dfrac{K(z/w)^{-(k-1)}}{z-w}$
$u(nT)e^{-\alpha nT}$	$\dfrac{z}{z-e^{-\alpha T}}$
$r(nT)$	$\dfrac{Tz}{(z-1)^2}$
$r(nT)e^{-\alpha nT}$	$\dfrac{Te^{-\alpha T}z}{(z-e^{-\alpha T})^2}$

- From the real-convolution theorem

$$\mathcal{Z} \sum_{k=-\infty}^{\infty} x_1(kT)x_2(nT - kT) = X_1(z)X_2(z)$$

Use of Real Convolution

- From the real-convolution theorem

$$\mathcal{Z}\sum_{k=-\infty}^{\infty} x_1(kT)x_2(nT-kT) = X_1(z)X_2(z)$$

- If we take the inverse z transform of both sides, we get

$$\sum_{k=-\infty}^{\infty} x_1(kT)_2(nT-kT) = \mathcal{Z}^{-1}[X_1(z)X_2(z)]$$

or

$$\mathcal{Z}^{-1}[X_1(z)X_2(z)] = \sum_{k=-\infty}^{\infty} x_1(kT)x_2(nT-kT)$$

Thus, if a z transform can be expressed as a *product* of two z transforms whose inverses are available, then performing the convolution summation will yield the desired inverse.

Example – Real Convolution

Find the inverse z transform of

$$X_3(z) = \frac{z}{(z-1)^2}$$

Solution We note that

$$X_3(z) = X_1(z)X_2(z)$$

where

$$X_1(z) = \frac{z}{z-1} \quad \text{and} \quad X_2(z) = \frac{1}{z-1}$$

...

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From the table of standard z transforms, we can write

$$x_1(nT) = u(nT) \quad \text{and} \quad x_2(nT) = u(nT - T)$$

Hence for $n \geq 0$, the real convolution yields

$$\begin{aligned} x_3(nT) = \sum_{k=-\infty}^{\infty} x_1(kT)x_2(nT-kT) &= \sum_{k=-\infty}^{\infty} u(kT)u(nT-T-kT) \\ &= \cdots + \overbrace{u(-T)u(nT)}^{k=-1} + \overbrace{u(0)u(nT-T)}^{k=0} + \overbrace{u(T)u(nT-2T)}^{k=1} + \cdots \\ &\quad + \overbrace{u(nT-T)u(0)}^{k=n-1} + \overbrace{u(nT)u(-T)}^{k=n} + \cdots \\ &= 0 + 1 + 1 + \cdots + 1 + 0 = n \end{aligned}$$

Example *Cont'd*

For $n < 0$, we have

$$x_3(nT) = \sum_{k=-\infty}^{\infty} u(kT)u(nT - T - kT)$$

$$= \cdots + \overbrace{u(-T)u(nT)}^{k=-1} + \overbrace{u(0)u(nT - T)}^{k=0} + \overbrace{u(T)u(nT - 2T)}^{k=1} + \cdots$$

$$+ \overbrace{u(nT - T)u(0)}^{k=n-1} + \overbrace{u(nT)u(-T)}^{k=n} + \cdots$$

and since all the terms are zero, we get

$$x_3(nT) = 0$$

(This result also follows from the initial value theorem.)

Example *Cont'd*

Summarizing, for $n \geq 0$,

$$x_3(nT) = n$$

and for $n < 0$,

$$x_3(nT) = 0$$

Therefore, for any value of n, we have

$$x_3(nT) = u(nT)n \quad \blacksquare$$

Example – Real Convolution

Using the real-convolution theorem, find the inverse z transforms of

$$X_3(z) = \frac{z}{(z-1)^3}$$

Solution For this example, we can write

$$X_1(z) = \frac{z}{(z-1)^2} \quad \text{and} \quad X_2(z) = \frac{1}{z-1}$$

and from the previous example, we have

$$x_1(nT) = u(nT)n \quad \text{and} \quad x_2(nT) = u(nT - T)$$

From the initial value theorem, for $n < 0$, we have

$$x_3(n) = 0$$

For $n \geq 0$, the convolution summation gives

$$\begin{aligned}
x_3(nT) &= \sum_{k=-\infty}^{\infty} ku(kT)u(nT - T - kT) \\
&= +\overbrace{0 \cdot [u(nT - T)]}^{k=0} + \overbrace{1 \cdot [u(nT - 2T)]}^{k=1} + \cdots \\
&\quad + \overbrace{(n-1)u(0)}^{k=n-1} + \overbrace{nu(-T)}^{k=n} \\
&= +0 + 1 + 2 + \cdots + n - 1 + 0 \\
&= \sum_{k=1}^{n-1} k
\end{aligned}$$

Example *Cont'd*

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Example *Cont'd*

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- The story goes that Gauss' mathematics teacher had something to attend to and wanted to keep his class busy. So he asked the class to find the sum:

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- As the teacher was getting ready to leave, Gauss shouted out "Sir, the answer is 4950!"

Example *Cont'd*

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- The story goes that Gauss' mathematics teacher had something to attend to and wanted to keep his class busy. So he asked the class to find the sum:

$$1 + 2 + 3 + \cdots 99$$

- As the teacher was getting ready to leave, Gauss shouted out "Sir, the answer is 4950!"
- "It's very simple, Sir, twice the sum is 100×99".

Example *Cont'd*

Gauss' reasoning was as follows:

$$\begin{array}{ccccc} 1 & 2 & 3 & \cdots & n-1 \\ n-1 & n-2 & n-3 & \cdots & 1 \\ --- & --- & --- & & --- \\ n & n & n & \cdots & n \end{array}$$

That is,

$$\sum_{k=1}^{n-1} k = \tfrac{1}{2} n(n-1)$$

Using this result, $x_3(nT)$ can be obtained as

$$x_3(nT) = \sum_{k=1}^{n-1} k = \tfrac{1}{2} u(nT) n(n-1) \quad \blacksquare$$

- Given a z transform $X(z) = N(z)/D(z)$, a series that converges in the outermost annulus of $X(z)$ can be readily obtained by arranging the numerator and denominator polynomials in descending powers of z and then performing polynomial division also known as *long division*.

Example – Long Division

Using long division, find the inverse z transform of

$$X(z) = \frac{-\frac{1}{4} + \frac{1}{2}z - \frac{1}{2}z^2 - \frac{7}{4}z^3 + 2z^4 + z^5}{-\frac{1}{4} + \frac{1}{4}z - z^2 + z^3}$$

Solution The numerator and denominator polynomials can be arranged in descending powers of z as

$$X(z) = \frac{z^5 + 2z^4 - \frac{7}{4}z^3 - \frac{1}{2}z^2 + \frac{1}{2}z - \frac{1}{4}}{z^3 - z^2 + \frac{1}{4}z - \frac{1}{4}}$$

Example *Cont'd*

$$z^2 + 3z + 1 + z^{-2} + z^{-3} + \tfrac{3}{4}z^{-4} + \cdots$$

$$z^3 - z^2 + \tfrac{1}{4}z - \tfrac{1}{4} \;\Big|\; z^5 + 2z^4 - \tfrac{7}{4}z^3 - \tfrac{1}{2}z^2 + \tfrac{1}{2}z - \tfrac{1}{4}$$

$$\mp z^5 \pm z^4 \mp \tfrac{1}{4}z^3 \pm \tfrac{1}{4}z^2$$

$$3z^4 - \tfrac{8}{4}z^3 - \tfrac{1}{4}z^2 + \tfrac{1}{2}z - \tfrac{1}{4}$$

$$\mp 3z^4 \pm 3z^3 \mp \tfrac{3}{4}z^2 \pm \tfrac{3}{4}z$$

$$z^3 - z^2 + \tfrac{5}{4}z - \tfrac{1}{4}$$

$$\mp z^3 \pm z^2 \mp \tfrac{1}{4}z \pm \tfrac{1}{4}$$

$$z$$

$$\mp z \pm 1 \mp \tfrac{1}{4}z^{-1} \pm \tfrac{1}{4}z^{-2}$$

$$1 - \tfrac{1}{4}z^{-1} + \tfrac{1}{4}z^{-2}$$

$$\mp 1 \pm z^{-1} \mp \tfrac{1}{4}z^{-2} \pm \tfrac{1}{4}z^{-3}$$

$$\tfrac{3}{4}z^{-1} + \tfrac{1}{4}z^{-3}$$

$$\vdots$$

Example *Cont'd*

Therefore,

$$X(z) = z^2 + 3z + 1 + z^{-2} + z^{-3} + \tfrac{3}{4}z^{-4} + \cdots \quad \blacksquare$$

i.e.,

$$x(-2T) = 1, \quad x(-T) = 3, \quad x(0) = 1, \quad x(T) = 0$$

$$x(2T) = 1, \quad x(3T) = 1, \quad x(4T) = \tfrac{3}{4}, \quad \ldots$$

- As illustrated by the previous example, the long-division approach readily yields any nonzero values of the signal for $n \leq 0$ but *does not yield* a closed-form solution.

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Use of Long Division *Cont'd*

- As illustrated by the previous example, the long-division approach readily yields any nonzero values of the signal for $n \leq 0$ but *does not yield* a closed-form solution.
- On the other hand, the general-inversion method yields a *closed-form* solution but presents certain difficulties in z transforms of two-sided signals because such z transforms have a higher-order pole at the origin whose residue is difficult to obtain.
- The inverses of such z transforms can be easily obtained in closed form by finding the values of the signal for $n \leq 0$ using *long division* and then applying the *general inversion method* to the remainder of the long division.

- Consider a z transform whose numerator degree exceeds the denominator degree of the form

$$X(z) = \frac{N(z)}{D(z)} = \frac{\sum_{i=0}^{M} a_i z^{M-i}}{\sum_{i=0}^{N} b_i z^{N-i}}$$

Use of Long Division *Cont'd*

- Consider a z transform whose numerator degree exceeds the denominator degree of the form

$$X(z) = \frac{N(z)}{D(z)} = \frac{\sum_{i=0}^{M} a_i z^{M-i}}{\sum_{i=0}^{N} b_i z^{N-i}}$$

- The first nonzero value of $x(nT)$ occurs at $n = (N - M)T$ according to the initial value theorem.

- Performing long division until the signal values $x[(N-M)T]$, $x[(N-M+1)T]$, ..., $x(0)$ are obtained, $X(z)$ can be expressed as

$$X(z) = \frac{N(z)}{D(z)} = Q(z) + R(z)$$

where

$$Q(z) = x[(N-M)T]z^{(M-N)} + x[(N-M+1)T]z^{(M-N-1)} + \cdots + x(0)$$

is the *quotient* polynomial and

$$R(z) = \frac{N'(z)}{D(z)}$$

is the *remainder* whose numerator degree is less than the denominator degree.

...

$$X(z) = \frac{N(z)}{D(z)} = Q(z) + R(z) \quad \text{where} \quad R(z) = \frac{N'(z)}{D(z)}$$

- Hence

$$\begin{aligned} x(nT) &= \mathcal{Z}^{-1} Q(z) + \mathcal{Z}^{-1} \frac{N'(z)}{D(z)} \\ &= x[(N-M)T] z^{(M-N)} + x[(N-M+1)T] z^{(M-N-1)} + \cdots \\ &\quad + x(0) + \mathcal{Z}^{-1} \frac{N'(z)}{D(z)} \end{aligned}$$

Since $\mathcal{Z}^{-1} \frac{N'(z)}{D(z)}$ represents a right-sided signal, it can be *easily* evaluated in *closed-form* by using the general inversion method.

Example – Long Division with General Inversion Method

Using long division along with the general inversion method, obtain a closed-form solution for the inverse z transform of

$$X(z) = \frac{-\frac{1}{4} + \frac{1}{2}z - \frac{1}{2}z^2 - \frac{7}{4}z^3 + 2z^4 + z^5}{-\frac{1}{4} + \frac{1}{4}z - z^2 + z^3}$$

Example *Cont'd*

Solution

$$
\begin{array}{r|l}
 & z^2 + 3z + 1 \\
\hline
z^3 - z^2 + \frac{1}{4}z - \frac{1}{4} & z^5 + 2z^4 - \frac{7}{4}z^3 - \frac{1}{2}z^2 + \frac{1}{2}z - \frac{1}{4} \\
 & \mp z^5 \pm z^4 \mp \frac{1}{4}z^3 \pm \frac{1}{4}z^2 \\
\cline{2-2}
 & 3z^4 - \frac{8}{4}z^3 - \frac{1}{4}z^2 + \frac{1}{2}z - \frac{1}{4} \\
 & \mp 3z^4 \pm 3z^3 \mp \frac{3}{4}z^2 \pm \frac{3}{4}z \\
\cline{2-2}
 & z^3 - z^2 + \frac{5}{4}z - \frac{1}{4} \\
 & \mp z^3 \pm z^2 \mp \frac{1}{4}z \pm \frac{1}{4} \\
\cline{2-2}
 & z
\end{array}
$$

Hence

$$X(z) = Q(z) + R(z) = z^2 + 3z + 1 + \frac{z}{z^3 - z^2 + \frac{1}{4}z - \frac{1}{4}}$$

Applying the inverse z transform, we have

$$\begin{aligned} x(nT) &= \mathcal{Z}^{-1}\left(z^2 + 3z + 1 + \frac{z}{z^3 - z^2 + \frac{1}{4}z - \frac{1}{4}}\right) \\ &= x(-2T)z^2 + x(-T)z + x(0) + \mathcal{Z}^{-1}R(z) \end{aligned}$$

where $x(-2T) = 1$, $x(-T) = 3$, $x(0) = 1$, and

$$R(z) = \frac{z}{z^3 - z^2 + \frac{1}{4}z - \frac{1}{4}} = \frac{z}{(z-1)(z+j\frac{1}{2})(z-j\frac{1}{2})}$$

The inverse z transform of $R(z)$ can now be obtained by using the general inversion method.

...

$$R(z) = \frac{z}{z^3 - z^2 + \frac{1}{4}z - \frac{1}{4}} = \frac{z}{(z-1)(z+j\frac{1}{2})(z-j\frac{1}{2})}$$

Since $-j\frac{1}{2} = \frac{1}{2}e^{-j\pi/2}$, the residues of $R(z)z^{n-1}$ can be obtained as

$$R_1 = \lim_{z\to 1} \frac{z^n}{(z^2+\frac{1}{4})} = \frac{1}{1+\frac{1}{4}} = \frac{4}{5}$$

$$R_2 = \lim_{z\to -j\frac{1}{2}} \frac{z^n}{(z-1)(z-j\frac{1}{2})} = \frac{\left(\frac{1}{2}\right)^n e^{-jn\pi/2}}{(-\frac{1}{2}+j)}$$

$$= \frac{2}{\sqrt{5}} \frac{\left(\frac{1}{2}\right)^n e^{-jn\pi/2}}{e^{j(\pi - tan^{-1}2)}} = \frac{2}{\sqrt{5}}\left(\frac{1}{2}\right)^n e^{-j(n\pi/2+\pi-tan^{-1}2)}$$

$$R_3 = R_2^* = \frac{2}{\sqrt{5}}\left(\frac{1}{2}\right)^n e^{j(n\pi/2+\pi-tan^{-1}2)}$$

Thus for $n \geq 1$, we have

$$\begin{aligned} R(z) &=) \; R_1 + R_2 + R_3 \\ &= \tfrac{4}{5} + \tfrac{4}{\sqrt{5}} \left(\tfrac{1}{2}\right)^n \tfrac{1}{2} \left[e^{j(n\pi/2+\pi-tan^{-1}2)} + e^{-j(n\pi/2+\pi-tan^{-1}2)} \right] \end{aligned}$$

Hence

$$r(nT) = \tfrac{4}{5} u(nT) + \tfrac{4}{\sqrt{5}} \left(\tfrac{1}{2}\right)^n \cos(n\pi/2 + \pi - tan^{-1}2)$$

Since $x(-2T) = 1$, $x(-T) = 3$, and $x(0) = 1$, the value of $x(nT)$ for any value of n is given by

$$\begin{aligned} x(nT) &= \delta(nT + 2T) + 3\delta(nT + T) + \delta(nT) \\ &\quad + u(nT - T)[\tfrac{4}{5} + \tfrac{4}{\sqrt{5}} \left(\tfrac{1}{2}\right)^n \cos(n\pi/2 + \pi - tan^{-1}2)] \quad \blacksquare \end{aligned}$$

Use of Partial Fractions

- If the degree of the numerator polynomial in $X(z)$ is equal to or less than the degree of the denominator polynomial and the poles are simple, the inverse of $X(z)$ can very quickly be obtained through the use of partial fractions.

Use of Partial Fractions

- If the degree of the numerator polynomial in $X(z)$ is equal to or less than the degree of the denominator polynomial and the poles are simple, the inverse of $X(z)$ can very quickly be obtained through the use of partial fractions.
- Two techniques are available, as detailed next.

Use of Partial Fractions, Technique I

- The function $X(z)/z$ can be expanded into partial fractions as

$$\frac{X(z)}{z} = \frac{R_0}{z} + \sum_{i=1}^{P} \frac{R_i}{z - p_i}$$

where P is the number of poles in $X(z)$ and

$$R_0 = \lim_{z \to 0} X(z) \qquad R_i = \text{res}_{\;z=p_i} \left[\frac{X(z)}{z} \right]$$

Use of Partial Fractions, Technique I

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- Hence

$$X(z) = R_0 + \sum_{i=1}^{P} \frac{R_i z}{z - p_i}$$

. . .

$$X(z) = R_0 + \sum_{i=1}^{P} \frac{R_i z}{z - p_i}$$

- Therefore,

$$x(nT) = \mathcal{Z}^{-1}\left(R_0 + \sum_{i=1}^{P} \frac{R_i z}{z - p_i}\right) = \mathcal{Z}^{-1} R_0 + \sum_{i=1}^{P} \mathcal{Z}^{-1} \frac{R_i z}{z - p_i}$$

and from the table of standard z transforms, we get

$$x(nT) = R_0 \delta(nT) + \sum_{i=1}^{P} u(nT) R_i p_i^n$$

Example – Partial Fractions Method

Using Technique I, find the inverse z transform of

$$X(z) = \frac{z}{z^2 + z + \frac{1}{2}}$$

Solution On expanding $X(z)/z$ into partial fractions, we get

$$\frac{X(z)}{z} = \frac{1}{z^2 + z + \frac{1}{2}} = \frac{1}{(z - p_1)(z - p_2)} = \frac{R_1}{z - p_1} + \frac{R_2}{z - p_2}$$

where $$p_1 = \frac{e^{j3\pi/4}}{\sqrt{2}} \quad \text{and} \quad p_2 = \frac{e^{-j3\pi/4}}{\sqrt{2}}$$

Thus we obtain

$$R_1 = \text{res}_{\,z=p_1}\left[\frac{X(z)}{z}\right] = -j \quad \text{and} \quad R_2 = \text{res}_{\,z=p_2}\left[\frac{X(z)}{z}\right] = j$$

Note: Complex conjugate poles give complex conjugate residues.

Example *Cont'd*

From the table of z transforms, we can now obtain

$$\begin{aligned} x(nT) &= u(nT)\big(-jp_1^n + jp_2^n\big) \\ &= \left(\tfrac{1}{2}\right)^{n/2} u(nT)\frac{1}{j}\big(e^{j3\pi n/4} - e^{-j3\pi n/4}\big) \\ &= 2\left(\tfrac{1}{2}\right)^{n/2} u(nT)\sin\frac{3\pi n}{4} \quad \blacksquare \end{aligned}$$

Use of Partial Fractions, Technique II

- An alternative approach is to expand $X(z)$ into partial fractions as

$$X(z) = R_0 + \sum_{i=1}^{P} \frac{R_i}{z - p_i}$$

where

$$R_0 = \lim_{z \to \infty} X(z) \qquad R_i = \text{res}_{\,z=p_i} X(z)$$

and P is the number of poles in $X(z)$.

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and P is the number of poles in $X(z)$.

- Thus

$$\begin{aligned} x(nT) &= \mathcal{Z}^{-1}\left[R_0 + \sum_{i=1}^{P} \frac{R_i}{z - p_i}\right] \\ &= \mathcal{Z}^{-1} R_0 + \sum_{i=1}^{P} \mathcal{Z}^{-1} \frac{R_i}{z - p_i} \end{aligned}$$

...

$$x(nT) = \mathcal{Z}^{-1} R_0 + \sum_{i=1}^{P} \mathcal{Z}^{-1} \frac{R_i}{z - p_i}$$

- Therefore, from Table 3.2, we obtain

$$X(nT) = R_0 \delta(nT) + \sum_{i=1}^{P} u(nT - T) R_i p_i^{n-1}$$

Example – Partial Fractions Method

Using Technique II, find the inverse z transform of

$$X(z) = \frac{z}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{4}\right)}$$

Solution $X(z)$ can be expressed as

$$X(z) = \frac{z}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{4}\right)} = R_0 + \frac{R_1}{z - \frac{1}{2}} + \frac{R_2}{z - \frac{1}{4}}$$

where

$$R_0 = \lim_{z\to\infty} X(z) = \lim_{z\to\infty} \frac{z}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{4}\right)} = \lim_{z\to\infty} \frac{1}{z} = 0$$

$$R_1 = \operatorname{res}_{z=\frac{1}{2}} X(z) = \left.\frac{z}{\left(z - \frac{1}{4}\right)}\right|_{z=\frac{1}{2}} = 2$$

. . .

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$$R_1 = \operatorname{res}_{z=\frac{1}{2}} X(z) = \left.\frac{z}{\left(z-\frac{1}{4}\right)}\right|_{z=\frac{1}{2}} = 2$$

and

$$R_2 = \operatorname{res}_{z=\frac{1}{4}} X(z) = \left.\frac{z}{\left(z-\frac{1}{2}\right)}\right|_{z=\frac{1}{4}} = -1$$

Hence

$$X(z) = \frac{2}{z-\frac{1}{2}} + \frac{-1}{z-\frac{1}{4}}$$

and from Table 3.2

$$x(nT) = 4u(nT-T)\left[\left(\tfrac{1}{2}\right)^n - \left(\tfrac{1}{4}\right)^n\right] \quad \blacksquare$$

- The partial fraction method is based on the assumption that the denominator degree of the z transform is equal to or greater than the numerator degree.

Use of Partial Fractions *Cont'd*

- The partial fraction method is based on the assumption that the denominator degree of the z transform is equal to or greater than the numerator degree.
- If this is not the case, then through long division the z transform can be expressed as

$$X(z) = \frac{N(z)}{D(z)} = Q(z) + R(z)$$

where

$$Q(z) = x[(N-M)T]z^{(M-N)} + x[(N-M+1)T]z^{(M-N-1)} + \cdots + x(0)$$

is the *quotient* polynomial and

$$R(z) = \frac{N'(z)}{D(z)}$$

is the *remainder* polynomial whose denominator degree is greater than the numerator degree.

Important Notes

- Given a z transform $X(z)$, a partial fraction expansion can be obtained through the following steps:
  - represent the residues by variables,
  - generate a system of simultaneous equations, and then
  - solve the system of equations for the residues.

- For example, if

$$X(z) = \frac{z^2 - 2}{(z-1)(z-2)} \qquad \text{(A)}$$

we can write

$$\begin{aligned} X(z) &= R_0 + \frac{R_1}{z-1} + \frac{R_2}{z-2} \\ &= \frac{R_0(z-1)(z-2) + R_1 z - 2R_1 + R_2 z - R_2}{(z-1)(z-2)} \\ &= \frac{R_0(z^2 - 3z + 2) + R_1 z - 2R_1 + R_2 z - R_2}{(z-1)(z-2)} \\ &= \frac{R_0 z^2 - 3R_0 z + 2R_0 + R_1 z - 2R_1 + R_2 z - R_2}{(z-1)(z-2)} \\ &= \frac{R_0 z^2 + (R_1 + R_2 - 3R_0)z + 2R_0 - 2R_1 - R_2}{(z-1)(z-2)} \qquad \text{(B)} \end{aligned}$$

- By equating equal powers of z in Eqs. (A) and (B), we get

$$\begin{aligned} z^2: &\quad R_0 = 1 \\ z^1: &\quad R_1 + R_2 - 3R_0 = 0 \\ z^0: &\quad 2R_0 - 2R_1 - R_2 = -2 \end{aligned}$$

Important Notes *Cont'd*

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- Solving this system of equations would give the correct solution as

$$R_0 = 1, \quad R_1 = 1, \quad R_2 = 2$$

Important Notes *Cont'd*

- For a z transform with six poles, a set of 6 simultaneous equations with 6 unknowns would need to be solved.

Important Notes *Cont'd*

- For a z transform with six poles, a set of 6 simultaneous equations with 6 unknowns would need to be solved.
- Obviously, this is a very *inefficient method* and it should definitely be avoided.

Important Notes *Cont'd*

- The quick solution for this example is easily obtained by evaluating the residues individually, as follows:

$$R_0 = \left.\frac{z^2 - 2}{(z-1)(z-2)}\right|_{z=\infty} = 1, \quad R_1 = \left.\frac{z^2 - 2}{(z-2)}\right|_{z=1} = 1$$

$$R_2 = \left.\frac{z^2 - 2}{(z-1)}\right|_{z=2} = 2$$

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$$R_2 = \left.\frac{z^2-2}{(z-1)}\right|_{z=2} = 2$$

- Hence

$$\begin{aligned} X(z) &= \frac{z^2-2}{(z-1)(z-2)} = R_0 + \frac{R_1}{z-1} + \frac{R_2}{z-2} \\ &= 1 + \frac{1}{z-1} + \frac{2}{z-2} \end{aligned}$$

Important Notes *Cont'd*

- In the partial-fraction method, constant R_0 must always be included although it may sometimes be found to be zero.

Important Notes *Cont'd*

- In the partial-fraction method, constant R_0 must always be included although it may sometimes be found to be zero.
- For example, if R_0 were omitted in the partial-fraction expansion

$$X(z) = \frac{z^2 - 2}{(z-1)(z-2)} = R_0 + \frac{R_1}{z-1} + \frac{R_2}{z-2}$$

then the right-hand side would assume the form

$$\frac{R_1}{z-1} + \frac{R_2}{z-2} = \frac{(R_1 + R_2)z - (2R_1 + R_2)}{(z-1)(z-2)}$$

which cannot represent the given function whatever the values of R_1 and R_2!

Important Notes *Cont'd*

- By the way, you can always check your work by combining the partial fractions back into a function, as you can check a division by multiplying.

This slide concludes the presentation.
Thank you for your attention.