Chapter 5
THE APPLICATION OF THE Z TRANSFORM
5.1 Introduction
5.2 The Discrete-Time Transfer Function

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Introduction

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- It can be used to obtain the time-domain response of a system to any excitation or its frequency-domain response.

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- The discrete-time transfer function plays the same key role as the continuous-time transfer function in an analog system.
- It can be used to obtain the time-domain response of a system to any excitation or its frequency-domain response.
- In this presentation, the definition, derivation, and properties of the discrete-time transfer function are examined.

Discrete-Time Transfer Function

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- Consider a linear time-invariant discrete-time system and let
  - $x(nT)$ be the excitation (or input)
  - $y(nT)$ be the response (or output)
  - $h(nT)$ be the impulse response

Discrete-Time Transfer Function *Cont'd*

- The convolution summation gives

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- From the real-convolution theorem (see Chap. 3), we have

$$\mathcal{Z}y(nT) = \mathcal{Z}x(nT)\mathcal{Z}h(nT)$$

- Therefore,

$$\frac{Y(z)}{X(z)} = H(z)$$

In effect, *the transfer function is also the z transform of the impulse response of the system*.

Derivation of Transfer Function from Difference Eqn.

- A noncausal, linear, time-invariant, recursive discrete-time system can be represented by the difference equation

$$y(nT) = \sum_{i=-M}^{N} a_i x(nT - iT) - \sum_{i=1}^{N} b_i y(nT - iT)$$

where M and N are positive integers.

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- The z transform gives

$$Y(z) = \mathcal{Z} y(nT) = \mathcal{Z} \sum_{i=-M}^{N} a_i x(nT - iT) - \mathcal{Z} \sum_{i=1}^{N} b_i y(nT - iT)$$

...

$$Y(z) = \mathcal{Z} \sum_{i=-M}^{N} a_i x(nT - iT) - \mathcal{Z} \sum_{i=1}^{N} b_i y(nT - iT)$$

Using the *linearity and time-shifting theorems* of the z transform, we get

$$\begin{aligned} Y(z) &= \sum_{i=-M}^{N} a_i z^{-i} \mathcal{Z} x(nT) - \sum_{i=1}^{N} b_i z^{-i} \mathcal{Z} y(nT) \\ &= \sum_{i=-M}^{N} a_i z^{-i} X(z) - \sum_{i=1}^{N} b_i z^{-i} Y(z) \end{aligned}$$

Transfer Function from Difference Equation *Cont'd*

. . .

$$Y(z) = \sum_{i=-M}^{N} a_i z^{-i} X(z) - \sum_{i=1}^{N} b_i z^{-i} Y(z)$$

Solving for $Y(z)/X(z)$ and then multiplying the numerator and denominator polynomials by z^N, we get

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{i=-M}^{N} a_i z^{-i}}{1 + \sum_{i=1}^{N} b_i z^{-i}} = \frac{\sum_{i=-M}^{N} a_i z^{N-i}}{z^N + \sum_{i=1}^{N} b_i z^{N-i}}$$

$$= \frac{a_{(-M)} z^{M+N} + a_{(-M+1)} z^{M+N-1} + \cdots + a_N}{z^N + b_1 z^{N-1} + \cdots + b_N}$$

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If $M = N = 2$, we have

$$H(z) = \frac{N(z)}{D(z)} = \frac{a_{(-2)}z^4 + a_{(-1)}z^3 + a_0 z^2 + a_1 z + a_2}{z^2 + b_1 z + b_2}$$

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Note: In noncausal systems, the degree of the numerator polynomial exceeds the degree of the denominator polynomial.

Transfer Function from Difference Equation *Cont'd*

- For a causal system, $M = 0$ and hence

$$H(z) = \frac{a_{(-M)} z^{M+N} + a_{(-M+1)} z^{M+N-1} + \cdots + a_N}{z^N + b_1 z^{N-1} + \cdots + b_N}$$
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- Since some of the numerator coefficients can be zero, we conclude that *in causal recursive systems the numerator degree is equal to or less than the denominator degree*.

Representation by Zero-Pole Plots

- By factorizing the numerator and denominator polynomials, the transfer function of a noncausal system can be expressed as

$$H(z) = \frac{N(z)}{D(z)} = \frac{H_0 \prod_{i=1}^{Z} (z - z_i)^{m_i}}{\prod_{i=1}^{P} (z - p_i)^{n_i}}$$

  where
  - $z_1, z_2, \ldots, z_Z$ are the zeros of $H(z)$
  - $p_1, p_2, \ldots, p_P$ are the poles of $H(z)$
  - m_i is the order of zero z_i
  - n_i is the order of pole p_i
  - $M + N = \sum_{i=1}^{Z} m_i$ is the order of $N(z)$
  - $N = \sum_{i=1}^{P} n_i$ is the order of $D(z)$
  - H_0 is a multiplier constant

...

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- The order of a discrete-time transfer function is the *order of $N(z)$ or $D(z)$, whichever is larger*, i.e., $M + N$ if $M > 0$ or N if $M = 0$.

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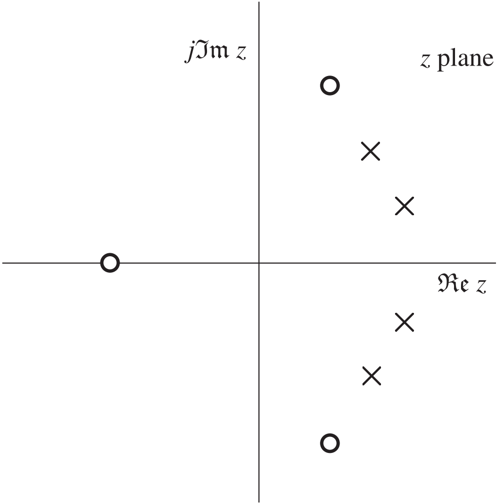
- Discrete-time systems can be represented by zero-pole plots.

Representation by Zero-Pole Plots *Cont'd*



Transfer Function in Nonrecursive Systems

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- Hence the transfer function assumes the form

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$$= \frac{\sum_{i=0}^{N} a_i z^{N-i}}{z^N}$$

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- Evidently, *the poles of nonrecursive systems are all located at the origin* of the z plane.

Derivation of Transfer Function from a Network

- The unit delay, adder, and multiplier are characterized by the equations

$$y(nT) = x(nT - T), \quad y(nT) = \sum_{i=1}^{K} x_i(nT), \quad y(nT) = mx(nT)$$

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- Hence if we apply the z transform, we get

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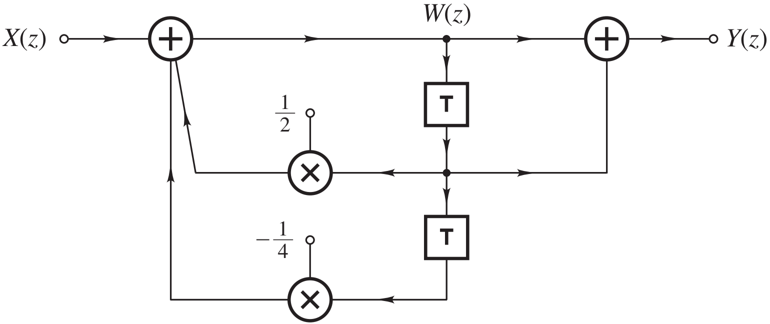
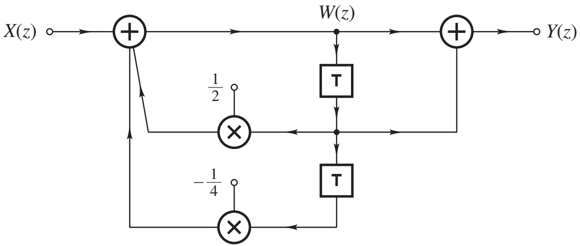
- By using these relations, *$H(z)$ can be obtained directly from a network representation*.

Example

Find the transfer function of the system:





Solution By inspection

$$W(z) = X(z) + \tfrac{1}{2}z^{-1}W(z) - \tfrac{1}{4}z^{-2}W(z)$$

and

$$Y(z) = W(z) + z^{-1}W(z)$$

...

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Hence

$$W(z) = \frac{X(z)}{1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}}, \qquad Y(z) = (1 + z^{-1})W(z)$$

If we eliminate $W(z)$ in the right-hand equation, we obtain

$$H(z) = \frac{z(z+1)}{z^2 - \frac{1}{2}z + \frac{1}{4}} \quad \blacksquare$$

Derivation from a State-Space Representation

- A discrete-time system can be represented by the state-space representation

$$\mathbf{q}(nT + T) = \mathbf{A}\mathbf{q}(nT) + \mathbf{b}x(nT) \qquad \text{(A)}$$

$$y(nT) = \mathbf{c}^T\mathbf{q}(nT) + dx(nT) \qquad \text{(B)}$$

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- Applying the z transform to Eq. (A), we get

$$\mathcal{Z}\mathbf{q}(nT + T) = \mathbf{A}\mathcal{Z}\mathbf{q}(nT) + \mathbf{b}\mathcal{Z}x(nT)$$

or

$$z\mathbf{Q}(z) = \mathbf{A}\mathbf{Q}(z) + \mathbf{b}X(z)$$

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-

Hence

$$z\mathbf{I}\mathbf{Q}(z) = \mathbf{A}\mathbf{Q}(z) + \mathbf{b}X(z)$$

or

$$\mathbf{Q}(z) = (z\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}X(z) \qquad \text{(C)}$$

where $\mathbf{I}$ is the identity matrix.

...

$$\mathbf{q}(nT+T) = \mathbf{A}\mathbf{q}(nT) + \mathbf{b}x(nT) \quad \text{(A)}$$

$$y(nT) = \mathbf{c}^T\mathbf{q}(nT) + dx(nT) \quad \text{(B)}$$

or

$$\mathbf{Q}(z) = (z\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}X(z) \quad \text{(C)}$$

- Now from Eq. (B)

$$Y(z) = \mathbf{c}^T\mathbf{Q}(z) + dX(z) \quad \text{(D)}$$

...

$$\mathbf{q}(nT + T) = \mathbf{A}\mathbf{q}(nT) + \mathbf{b}x(nT) \qquad \text{(A)}$$

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or

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- Now from Eq. (B)

$$Y(z) = \mathbf{c}^T\mathbf{Q}(z) + dX(z) \qquad \text{(D)}$$

- If we now eliminate $\mathbf{Q}(z)$ using Eq. (D), we have

$$H(z) = \frac{Y(z)}{X(z)} = \frac{N(z)}{D(z)} = \mathbf{c}^T(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + d \quad \blacksquare$$

where $N(z)$ and $D(z)$ are polynomials in z.

This slide concludes the presentation.
Thank you for your attention.