Chapter 5
THE APPLICATION OF THE Z TRANSFORM
5.3 Stability

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Stability

- A discrete-time system is stable if and only if its impulse response is absolutely summable, i.e.,

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- Since the transfer function is the z transform of the impulse response, we expect the stability of the filter to depend critically on the transfer function.

  It actually depends *exclusively* on the positions of the poles.

Stability *Cont'd*

- Consider a causal recursive system characterized by the transfer function

$$H(z) = \frac{N(z)}{D(z)} = \frac{H_0 \prod_{i=1}^{M} (z - z_i)^{m_i}}{\prod_{i=1}^{N} (z - p_i)^{n_i}} \quad \text{where} \quad N \geq M$$

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- By using the residue theorem, we have

$$h(nT) = \begin{cases} R_0 + \sum_{i=1}^{N} \text{res}_{\ z=p_i} \left[H(z) z^{-1} \right] & \text{for } n = 0 \\ \sum_{i=1}^{N} \text{res}_{\ z=p_i} [H(z) z^{n-1}] & \text{for } n > 0 \end{cases}$$

where

$$R_0 = \text{res}_{\ z=0} \left[\frac{H(z)}{z} \right]$$

if $H(z)/z$ has a pole at the origin and $R_0 = 0$ otherwise.

Stability *Cont'd*

- If we assume that $H(z)$ has *simple* poles, i.e., $n_i = 1$ for $i = 1, 2, \ldots, N$, then the impulse response can be expressed as

$$h(nT) = \begin{cases} R_0 + \sum_{i=1}^{N} p_i^{-1} \text{res}_{\;z=p_i} H(z) & \text{for } n = 0 \\ \sum_{i=1}^{N} p_i^{n-1} \text{res}_{\;z=p_i} H(z) & \text{for } n > 0 \end{cases}$$

  where the ith term in the summations is the contribution to the impulse response due to pole p_i.

Stability *Cont'd*

...

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- If we let

$$p_i = r_i e^{j\psi_i}$$

then the impulse response can be expressed as

$$h(nT) = \begin{cases} h(0) \\ \sum_{i=1}^{N} r_i^{n-1} e^{j(n-1)\psi_i} \text{res}_{\,z=p_i} H(z) & \text{for } n > 0 \end{cases}$$

where

$$h(0) = R_0 + \sum_{i=1}^{N} r_i^{-1} e^{-j\psi_i} \text{res}_{\,z=p_i} H(z) \quad \text{for } n = 0$$

is finite.

Stability *Cont'd*

. . .

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- We can now write

$$\sum_{n=0}^{\infty} |h(nT)| = |h(0)| + \sum_{n=1}^{\infty} \left| \sum_{i=1}^{N} r_i^{n-1} e^{j(n-1)\psi_i} \text{res}_{\, z=p_i} H(z) \right|$$

Stability *Cont'd*

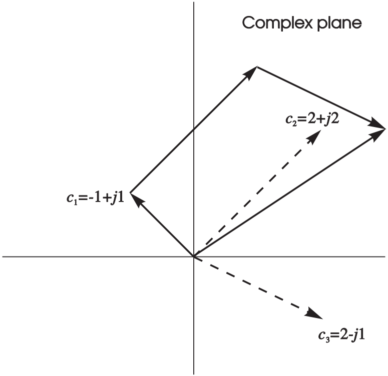
- We note that

$$\sum_{i=1}^{N} |i\text{th term}| \geq \left| \sum_{i=1}^{N} i\text{th term} \right|$$

Example

- The sum of the magnitudes of the complex numbers is

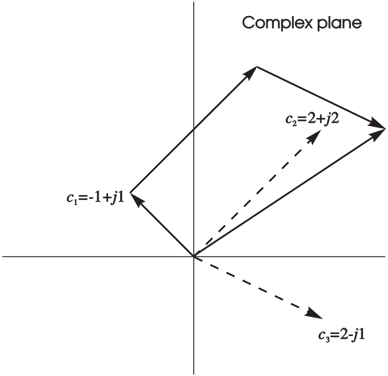
$$\sum_{i=1}^{3} |c_i| = |(-1+j1)|+|(2+j2)|+|(2-j1)| = \sqrt{2}+\sqrt{8}+\sqrt{5} = 6.479$$



Example *Cont'd*

- On the other hand, the magnitude of the sum of the complex numbers is given by

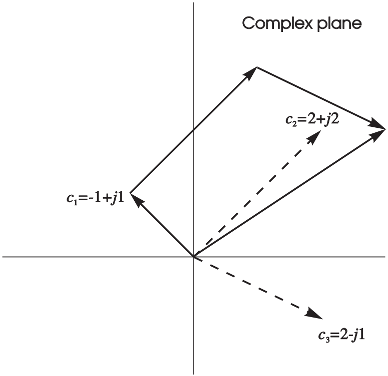
$$\left|\sum_{i=1}^{3} c_i\right| = |(-1+j1)+(2+j2)+(2-j1)| = |3+j2| = \sqrt{13} = 3.606$$



Example *Cont'd*

- Therefore,

$$\sum_{i=1}^{3} |c_i| \geq \left| \sum_{i=1}^{3} c_i \right|$$



Stability *Cont'd*

...

$$\sum_{n=0}^{\infty} |h(nT)| = |h(0)| + \sum_{n=1}^{\infty} \left| \sum_{i=1}^{N} r_i^{n-1} e^{j(n-1)\psi_i} \text{res}_{\,z=p_i} H(z) \right|$$

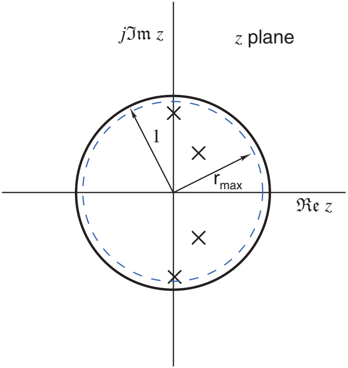
- Thus we can write

$$\begin{aligned}
\sum_{n=0}^{\infty} |h(nT)| &\le |h(0)| + \sum_{n=1}^{\infty} \sum_{i=1}^{N} \left| r_i^{n-1} e^{j(n-1)\psi_i} \text{res}_{\,z=p_i} H(z) \right| \\
&\le |h(0)| + \sum_{n=1}^{\infty} \sum_{i=1}^{N} \left| r_i^{n-1} \right| \left| e^{j(n-1)\psi_i} \right| \left| \text{res}_{\,z=p_i} H(z) \right| \\
&\le |h(0)| + \sum_{n=1}^{\infty} \sum_{i=1}^{N} r_i^{n-1} \left| \text{res}_{\,z=p_i} H(z) \right|
\end{aligned}$$

Stability *Cont'd*

- Let us assume that all the poles are inside the unit circle $|z| = 1$, i.e.,

$$r_i \le r_{\max} < 1 \quad \text{for} \quad i = 1, 2, \ldots, N$$



Stability *Cont'd*

- Now if p_k is a *simple* pole of some function $F(z)$, then function $(z - p_k)F(z)$ is analytic and, therefore, the residue of $F(z)$ at $z = p_k$ is finite.

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- Consequently, all the residues of $H(z)$ are finite and so

$$|\text{res}_{\,z=p_i} H(z)| \leq R_{\max} \quad \text{for} \quad i = 1, 2, \ldots, N$$

where $R_{\max}$ is a positive constant.

Stability *Cont'd*

- From the previous two slides

$$r_i \leq r_{\max} < 1 \quad \text{for} \quad i = 1, 2, \ldots, N$$

and

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- Therefore, we can write

$$\sum_{n=0}^{\infty} |h(nT)| \le |h(0)| + \sum_{n=1}^{\infty} \sum_{i=1}^{N} r_i^{n-1} |\text{res}_{\,z=p_i} H(z)|$$

$$\le |h(0)| + NR_{\max} \sum_{n=1}^{\infty} r_{\max}^{n-1}$$

Stability *Cont'd*

. . .

$$\sum_{n=0}^{\infty} |h(nT)| \le |h(0)| + NR_{\max} \sum_{n=1}^{\infty} r_{\max}^{n-1}$$

- The sum at the right-hand side is a geometric series with common ratio $r_{\max}$ and since we have assumed that $r_{\max} < 1$, the series converges.

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- We, therefore, conclude that

$$\sum_{n=0}^{\infty} |h(nT)| < \infty$$

- Summarizing, we have assumed that all the poles are inside the unit circle, i.e.,

$$r_i \leq r_{max} < 1 \quad \text{for} \quad i = 1, 2, \ldots, N$$

and demonstrated that in such a case the impulse response is absolutely summable, i.e.,

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- Therefore, we conclude that *if all the poles are inside the unit circle, the system is stable*.

Stability *Cont'd*

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- One more thing needs to be done in order to fully establish the role of the pole positions on the stability of the system.
- The condition established so far is a sufficient condition and one may, therefore, ask: Is it possible for a system to be stable if one or more poles are located *on or outside* the unit circle?

Stability *Cont'd*

- Let us assume that a single pole of $H(z)$, say pole p_k, is located on or outside the unit circle, i.e., $r_k \geq 1$.

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- Let us assume that a single pole of $H(z)$, say pole p_k, is located on or outside the unit circle, i.e., $r_k \geq 1$.
- In such a case, as $n \to \infty$ we have

$$h(nT) = \sum_{i=1}^{N} r_i^{n-1} e^{j(n-1)\psi_i} \text{res}_{\,z=p_i} H(z)$$
$$\approx r_k^{n-1} e^{j(n-1)\psi_k} \text{res}_{\,z=p_k} H(z)$$

since for a large value of n, $r_i^{n-1} \to 0$ for all $i \neq k$ for which $r_i < 1$ whereas r_k^{n-1} is unity or becomes very large since $r_k \geq 1$.

Stability *Cont'd*

- Thus

$$\sum_{n=0}^{\infty} |h(nT)| \approx \sum_{n=0}^{\infty} r_k^{n-1} \left| e^{j(n-1)\psi_i} \right| |\text{res}_{\ z=p_k} H(z)|$$

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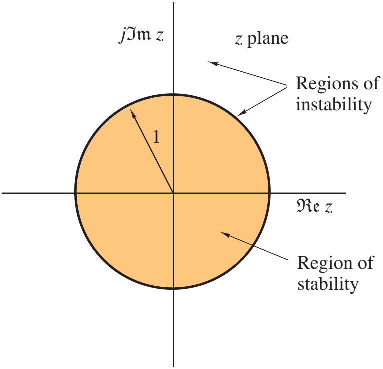
- Since $r_k \geq 1$, the sum at the right-hand side does not converge, i.e., the impulse response is not absolutely summable, i.e.,

$$\sum_{n=0}^{\infty} |h(nT)| \to \infty$$

and the system is *unstable*.

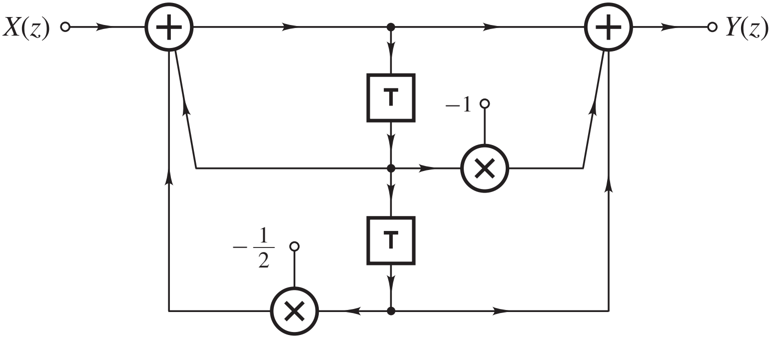
- Therefore, we conclude that *a discrete-time system is stable if and only if all its poles are inside the unit circle of the z plane.*



Note: Nonrecursive discrete-time systems are always stable since their poles are always located at the *origin* of the z plane.

Example

Check the following system for stability:



Solution The transfer function of the system can be easily obtained as

$$H(z) = \frac{z^2 - z + 1}{z^2 - z + 0.5} = \frac{z^2 - z + 1}{(z - p_1)(z - p_2)}$$

where

$$p_1,\ p_2 = \tfrac{1}{2} \ \pm j\tfrac{1}{2} = \tfrac{1}{\sqrt{2}} e^{\pm j\pi/4}$$

Since

$$|p_1|,\ |p_2| < 1$$

the system is stable. ■

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- Consider a system characterized by the transfer function

$$H(z) = \frac{N(z)}{D(z)}$$

where

$$N(z) = \sum_{i=0}^{M} a_i z^{M-i} \quad \text{and} \quad D(z) = \sum_{i=0}^{N} b_i z^{N-i}$$

Stability Criteria *Cont'd*

- As was demonstrated in previous slides, the stability of a discrete-time system can be determined by finding the poles of the transfer function, namely, the roots of the denominator polynomial $D(z)$.

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- For a second- or a third-order system this is easily done.
- For higher-order systems, we need to use a computer program that would evaluate the roots of a polynomial, for example, MATLAB.
- Alternatively, we can use one of several stability criteria.

- Common factors in $N(z)$ and $D(z)$ do not have anything to do with stability because they can be canceled out.

Stability Criteria *Cont'd*

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- For example, if $N(z)$ and $D(z)$ have a common factor $(z + w)$, then

$$H(z) = \frac{N(z)}{D(z)} = \frac{(z+w)N'(z)}{(z+w)D'(z)} = \frac{N'(z)}{D'(z)}$$

  In effect, the poles of $H(z)$ are the roots of $D'(z)$ and parameter w will not appear in the impulse response.

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- To test a transfer function

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  for common factors, an $(M + N) \times (M + N)$ matrix is constructed and its determinant is evaluated where M and N are the numerator and denominator degrees, respectively.

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- Hereafter, we assume that $N(z)$ and $D(z)$ do not have any common factors.

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  - Schur-Cohn-Fujiwara criterion (1925)
  - Jury-Marden criterion (1962)

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- For an Nth-order system, N matrices of dimensions ranging from 2×2 to $2N \times 2N$ are constructed using the coefficients of $D(z)$.
- The determinants of these matrices, say, $D_1, D_2, \ldots, D_N$, are computed and their signs are determined.
- The system is stable if and only if

$$D_k < 0 \quad \text{for} \;\; \text{odd} \;\; k \qquad \text{and} \qquad D_k > 0 \quad \text{for} \;\; \text{even} \;\; k$$

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$$D(z) = \sum_{i=0}^{N} b_i z^{N-i}$$

where $b_0 > 0$.

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  - The denominator of the transfer function is given by

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    where $b_0 > 0$.
  - The numerator and denominator polynomials of the transfer function, $N(z)$ and $D(z)$, do not have any common factors.
- The first assumption that $b_0 > 0$ simplifies the Jury-Marden stability criterion but it is not a limitation.

- If $b_0 < 0$ then we can multiply the numerator and denominator polynomials by -1 to get a positive b_0.

Jury-Marden Stability Criterion *Cont'd*

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Jury-Marden Stability Criterion *Cont'd*

- If $b_0 < 0$ then we can multiply the numerator and denominator polynomials by -1 to get a positive b_0.
- This does not change the pole positions.
- For example, if

$$H(z) = \frac{N(z)}{D(z)} = \frac{z^2 + 2z + 1}{-2z^2 + 0.8z - 0.4}$$

we can write

$$H(z) = \frac{(z^2 + 2z + 1)(-1)}{(-2z^2 + 0.8z - 0.4)(-1)} = \frac{-z^2 - 2z - 1}{2z^2 - 0.8z + 0.4} = \frac{N'(z)}{D'(z)}$$

where $D'(z)$ has a positive b_0.

Jury-Marden Stability Criterion *Cont'd*

Row	Coefficients							
1	b_0	b_1	b_2	b_3	$\cdots$	b_{N-2}	b_{N-1}	b_N
2	b_N	b_{N-1}	b_{N-2}	b_{N-3}	$\cdots$	b_2	b_1	b_0
3	c_0	c_1	c_2	$\cdots$	c_{N-3}	c_{N-2}	c_{N-1}	
4	c_{N-1}	c_{N-2}	c_{N-3}	$\cdots$	c_2	c_1	c_0	
5	d_0	d_1	d_2	$\cdots$	d_{N-3}	d_{N-2}		
6	d_{N-2}	d_{N-3}	d_{N-4}	$\cdots$	d_1	d_0		
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$			
2N - 3	r_0	r_1	r_2					

where

$$c_i = \begin{vmatrix} b_i & b_N \\ b_{N-i} & b_0 \end{vmatrix} = \begin{vmatrix} b_0 & b_{N-i} \\ b_N & b_i \end{vmatrix} \quad \text{for} \quad 0, 1, \ldots, N-1$$

$$d_i = \begin{vmatrix} c_i & c_{N-1} \\ c_{N-1-i} & c_0 \end{vmatrix} = \begin{vmatrix} c_0 & c_{N-1-i} \\ c_{N-1} & c_i \end{vmatrix} \quad \text{for} \quad 0, 1, \ldots, N-2$$

Jury-Marden Stability Criterion *Cont'd*

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  (i)

$$D(1) > 0$$

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  (i)
  $$D(1) > 0$$

  (ii)
  $$(-1)^N D(-1) > 0$$

Jury-Marden Stability Criterion *Cont'd*

- The Jury-Marden stability criterion states that polynomial $D(z)$ has roots inside the unit circle of the z plane (i.e., the filter is stable) if and only if the following conditions are satisfied:

(i)

$$D(1) > 0$$

(ii)

$$(-1)^N D(-1) > 0$$

(iii)

$$\begin{aligned} b_0 &> |b_N| \\ |c_0| &> |c_{N-1}| \\ |d_0| &> |d_{N-2}| \\ &\vdots \\ |r_0| &> |r_2| \end{aligned}$$

Example

A discrete-time system is characterized by the transfer function

$$H(z) = \frac{z^4}{4z^4 + 3z^3 + 2z^2 + z + 1}$$

Check the filter for stability.

Solution The denominator polynomial of the transfer function is given by

$$D(z) = 4z^4 + 3z^3 + 2z^2 + z + 1$$

Since

$$D(1) = 11 > 0 \quad \text{and} \quad (-1)^4 D(-1) = 3 > 0$$

conditions (i) and (ii) of the test are satisfied.

Example *Cont'd*

Jury-Marden array:

Row	Coefficients				
1	4	3	2	1	1
2	1	1	2	3	4
3	15	11	6	1	
4	1	6	11	15	
5	224	159	79		

Since

$$b_0 > |b_4|, \quad |c_0| > |c_3|, \quad |d_0| > |d_2|$$

condition (iii) is also satisfied and the *filter is stable*. ∎

Example

A discrete-time system is characterized by the transfer function

$$H(z) = \frac{z^2 + 2z + 1}{z^4 + 6z^3 + 3z^2 + 4z + 5}$$

Check the filter for stability.

Solution The denominator polynomial of the transfer function is given

$$D(z) = z^4 + 6z^3 + 3z^2 + 4z + 5$$

In this example,

$$(-1)^4 D(-1) = -1$$

Therefore, condition (ii) of the test is violated and the *filter is unstable*. ■

Note: Note that there is no need to construct the Jury-Marden array! Violating only one of the conditions is enough to demonstrate that the filter is unstable.

This slide concludes the presentation.
Thank you for your attention.