# Chapter 5
# THE APPLICATION OF THE $Z$ TRANSFORM
## 5.5.5 Frequency Response of Digital Filters


Victoria, BC, Canada
Email: aantoniou@ieee.org

July 14, 2018

# Introduction

- In Sec. 5.5, it is shown that the steady-state response of a *stable* $N$th-order discrete-time system to a sinusoidal signal

$$x(nT) = u(nT)\sin \omega nT$$

is another sinusoidal signal of the form

$$\lim_{nT\to\infty} y(nT) = \tilde{y}(nT) = M(\omega)\sin[\omega nT + \theta(\omega)]$$

# Introduction

- In Sec. 5.5, it is shown that the steady-state response of a *stable* $N$th-order discrete-time system to a sinusoidal signal

$$x(nT) = u(nT)\sin \omega nT$$

is another sinusoidal signal of the form

$$\lim_{nT\to\infty} y(nT) = \tilde{y}(nT) = M(\omega)\sin[\omega nT + \theta(\omega)]$$

- The quantities

$$M(\omega) = |H(e^{j\omega T})| \quad \text{and} \quad \theta(\omega) = \arg H(e^{j\omega T})$$

define the *amplitude response* and *phase response*, respectively, and

$$H(z)\Big|_{z=e^{j\omega T}} = H(e^{j\omega T}) = M(\omega)e^{j\theta(\omega)}$$

defines the *frequency response*.

# Introduction *Cont'd*

- It was also shown that the frequency response of a discrete-time system is periodic with period $\omega_s$.

- It was also shown that the frequency response of a discrete-time system is periodic with period $\omega_s$.
- Thus the frequency response is completely specified if it is known over the frequency range $-\omega_s/2 < \omega \leq \omega_s/2$.

  This frequency range is known as the *baseband*.

- It was also shown that the frequency response of a discrete-time system is periodic with period $\omega_s$.
- Thus the frequency response is completely specified if it is known over the frequency range $-\omega_s/2 < \omega \leq \omega_s/2$.

  This frequency range is known as the *baseband*.
- It can be easily shown that the amplitude response is an *even* function and the phase response is an *odd* function of $\omega$, i.e.,

$$M(-\omega) = M(\omega) \quad \text{and} \quad \theta(-\omega) = -\theta(\omega)$$

# Introduction *Cont'd*

- It was also shown that the frequency response of a discrete-time system is periodic with period $\omega_s$.
- Thus the frequency response is completely specified if it is known over the frequency range $-\omega_s/2 < \omega \leq \omega_s/2$.

  This frequency range is known as the *baseband*.
- It can be easily shown that the amplitude response is an *even* function and the phase response is an *odd* function of $\omega$, i.e.,

$$M(-\omega) = M(\omega) \quad \text{and} \quad \theta(-\omega) = -\theta(\omega)$$

- Therefore, the frequency response is completely specified if it is known over the positive half of the baseband, i.e., $0 \leq \omega \leq \omega_s/2$.

# Introduction *Cont'd*

- In Chap. 1, a number of different types of filters have been identified on the basis of which frequencies are passed or rejected, e.g., lowpass or highpass filters.

# Introduction *Cont'd*

- In Chap. 1, a number of different types of filters have been identified on the basis of which frequencies are passed or rejected, e.g., lowpass or highpass filters.
- In the analog world, a lowpass filter will pass low frequencies in the range $0 \leq \omega < \omega_c$ and reject high frequencies in the range $\omega_c < \omega < \infty$ where $\omega_c$ is called the *cutoff* frequency.

# Introduction *Cont'd*

- In Chap. 1, a number of different types of filters have been identified on the basis of which frequencies are passed or rejected, e.g., lowpass or highpass filters.
- In the analog world, a lowpass filter will pass low frequencies in the range $0 \leq \omega < \omega_c$ and reject high frequencies in the range $\omega_c < \omega < \infty$ where $\omega_c$ is called the *cutoff* frequency.
- On the other hand, an analog highpass filter will pass high frequencies in the range $\omega_c \leq \omega < \infty$ and reject low frequencies in the range $0 < \omega \leq \omega_c$.

- In the digital world, the filter is classified on the basis of its amplitude response with respect to the positive half of the baseband.

- In the digital world, the filter is classified on the basis of its amplitude response with respect to the positive half of the baseband.
- Thus a digital *lowpass* filter will pass low frequencies in the range $0 \leq \omega < \omega_c$ and reject high frequencies in the range $\omega_c < \omega < \omega_s/2$ where $\omega_c$ is the *cutoff* frequency, as in the case of analog filters.

- In the digital world, the filter is classified on the basis of its amplitude response with respect to the positive half of the baseband.
- Thus a digital *lowpass* filter will pass low frequencies in the range $0 \leq \omega < \omega_c$ and reject high frequencies in the range $\omega_c < \omega < \omega_s/2$ where $\omega_c$ is the *cutoff* frequency, as in the case of analog filters.
- A digital *highpass* filter will pass high frequencies in the range $\omega_c \leq \omega < \omega_s/2$ and reject low frequencies in the range $0 < \omega \leq \omega_c$.

- A digital *bandpass* filter will pass midband frequencies in the range $\omega_{c1} \leq \omega < \omega_{c2}$ and reject low frequencies in the range $0 < \omega \leq \omega_{c1}$ and high frequencies in the range $\omega_{c2} \leq \omega < \omega_s/2$ where $\omega_{c1}$ and $\omega_{c2}$ are said to be the *lower and upper* cutoff frequencies, respectively.

- A digital *bandpass* filter will pass midband frequencies in the range $\omega_{c1} \leq \omega < \omega_{c2}$ and reject low frequencies in the range $0 < \omega \leq \omega_{c1}$ and high frequencies in the range $\omega_{c2} \leq \omega < \omega_s/2$ where $\omega_{c1}$ and $\omega_{c2}$ are said to be the *lower and upper* cutoff frequencies, respectively.
- On the other hand, a digital *bandstop* filter will reject midband frequencies in the range $\omega_{c1} \leq \omega < \omega_{c2}$ and pass low frequencies in the range $0 < \omega \leq \omega_{c1}$ and high frequencies in the range $\omega_{c2} \leq \omega < \omega_s/2$.

# Introduction *Cont'd*

- A digital *bandpass* filter will pass midband frequencies in the range $\omega_{c1} \leq \omega < \omega_{c2}$ and reject low frequencies in the range $0 < \omega \leq \omega_{c1}$ and high frequencies in the range $\omega_{c2} \leq \omega < \omega_s/2$ where $\omega_{c1}$ and $\omega_{c2}$ are said to be the *lower and upper* cutoff frequencies, respectively.
- On the other hand, a digital *bandstop* filter will reject midband frequencies in the range $\omega_{c1} \leq \omega < \omega_{c2}$ and pass low frequencies in the range $0 < \omega \leq \omega_{c1}$ and high frequencies in the range $\omega_{c2} \leq \omega < \omega_s/2$.
- In other words, the upper edge of the baseband in digital systems is analogous to infinite frequency in analog systems.

- An arbitrary transfer function $H(z)$ can be expressed in terms of its magnitude and angle as

$$H(z) = |H(z)|e^{j \arg H(z)}$$

# Introduction *Cont'd*

- An arbitrary transfer function $H(z)$ can be expressed in terms of its magnitude and angle as

$$H(z) = |H(z)|e^{j \arg H(z)}$$

- If $z$ is a complex variable of the form $z = \text{Re } z + j\text{Im } z$, then

$$|H(z)| = |\text{Re } H(z) + j\text{Im } H(z)|$$

and

$$\arg H(z) = \tan^{-1} \frac{\text{Im } H(z)}{\text{Re } H(z)}$$

# Introduction *Cont'd*

- An arbitrary transfer function $H(z)$ can be expressed in terms of its magnitude and angle as

$$H(z) = |H(z)|e^{j \arg H(z)}$$

- If $z$ is a complex variable of the form $z = \text{Re } z + j\text{Im } z$, then

$$|H(z)| = |\text{Re } H(z) + j\text{Im } H(z)|$$

  and

$$\arg H(z) = \tan^{-1} \frac{\text{Im } H(z)}{\text{Re } H(z)}$$

- These quantities represent *surfaces* over the $z$ plane, which can be represented by 3-dimensional plots.

- An arbitrary transfer function $H(z)$ can be expressed in terms of its magnitude and angle as

$$H(z) = |H(z)|e^{j \arg H(z)}$$

- If $z$ is a complex variable of the form $z = \text{Re } z + j\text{Im } z$, then

$$|H(z)| = |\text{Re } H(z) + j\text{Im } H(z)|$$

and

$$\arg H(z) = \tan^{-1} \frac{\text{Im } H(z)}{\text{Re } H(z)}$$

- These quantities represent *surfaces* over the $z$ plane, which can be represented by 3-dimensional plots.

*Note:* The magnitude function $|H(z)|$ is, of course, a nonnegative quantity but $\arg H(z)$ can be positive or negative.

# Introduction *Cont'd*

- If we let $z = e^{j\omega T}$, i.e., if $z$ assumes values on the *unit circle* $|z| = 1$, then *3-D plots* of the form
  - $|H(e^{j\omega T})|$ versus $e^{j\omega T}$ and $\arg H(e^{j\omega T})$ versus $e^{j\omega T}$

  can be constructed which represent the amplitude and phase responses.

- If we let $z = e^{j\omega T}$, i.e., if $z$ assumes values on the *unit circle* $|z| = 1$, then *3-D plots* of the form
  - $|H(e^{j\omega T})|$ versus $e^{j\omega T}$ and $\arg H(e^{j\omega T})$ versus $e^{j\omega T}$

  can be constructed which represent the amplitude and phase responses.

- These 3-D plots are, of course, subsets of the plots
  - $|H(z)|$ versus $z$ and $arg(z)$ versus $z$.

# Introduction *Cont'd*

- If we let $z = e^{j\omega T}$, i.e., if $z$ assumes values on the *unit circle* $|z| = 1$, then *3-D plots* of the form
  - $|H(e^{j\omega T})|$ versus $e^{j\omega T}$ and $\arg H(e^{j\omega T})$ versus $e^{j\omega T}$

  can be constructed which represent the amplitude and phase responses.

- These 3-D plots are, of course, subsets of the plots
  - $|H(z)|$ versus $z$ and $arg(z)$ versus $z$.

- From these 3-D plots, *2-D plots* of the form
  - $M(\omega)$ versus $\omega$ and $\theta(\omega)$ versus $\omega$

  can be constructed, which represent the amplitude and phase responses.

# Introduction *Cont'd*

- In this presentation, we explore the various types of geometrical representations that are associated with
  - the transfer function,
  - the amplitude response, and
  - the phase response.

- In this presentation, we explore the various types of geometrical representations that are associated with
  - the transfer function,
  - the amplitude response, and
  - the phase response.
- The various representations are illustrated in terms of specific transfer functions for
  - a lowpass recursive filter,
  - a lowpass nonrecursive filter, and
  - a bandpass recursive filter.

# Geometrical Representations

- If

$$H(z) = \frac{N(z)}{D(z)} = \frac{H_0 \prod_{i=1}^{N}(z - z_i)}{\prod_{i=1}^{N}(z - p_i)}$$

then the zeros $z_1, z_2, \ldots$ of $H(z)$ will show up as dimples in the surface $|H(z)|$ whereas the poles $p_1, p_2, \ldots$ will show up as spikes.

# Geometrical Representations

- If

$$H(z) = \frac{N(z)}{D(z)} = \frac{H_0 \prod_{i=1}^{N}(z - z_i)}{\prod_{i=1}^{N}(z - p_i)}$$

then the zeros $z_1, z_2, \ldots$ of $H(z)$ will show up as dimples in the surface $|H(z)|$ whereas the poles $p_1, p_2, \ldots$ will show up as spikes.

- The slides that follow will illustrate the various geometrical representations that are associated with the transfer function, amplitude response and phase response, e.g.,
  - zero-pole plot
  - 3-D plots of $|H(z)|$ and $\arg H(z)$ versus $z = \text{Re } z + j\text{Im } z$
  - 3-D plots of $|H(e^{j\omega T})|$ and $\arg H(e^{j\omega T})$ versus $z = e^{j\omega T}$
  - 2-D plots of $M(\omega) = |H(e^{j\omega T})|$ and $\theta(\omega) = \arg H(e^{j\omega T})$ versus $\omega$

- Consider a fourth-order lowpass digital filter that has the following transfer function

$$H(z) = H_0 \prod_{i=1}^{2} H_i(z) \quad \text{where} \quad H_i(z) = \frac{a_{0i} + a_{1i}z + z^2}{b_{0i} + b_{1i}z + z^2}$$

with

$$\begin{aligned}
H_0 &= 6.351486E-02 \\
a_{01} &= 1.0, \quad a_{11} = 1.494070 \\
b_{01} &= 5.115041E-01, \quad b_{11} = -1.015631 \\
a_{02} &= 1.0, \quad a_{12} = 4.188149E-01 \\
b_{02} &= 8.839638E-01 \quad b_{12} = -3.548538E-01
\end{aligned}$$

# Lowpass Filter *Cont'd*

- The transfer function can be expressed in terms of its zeros and poles as

$$H(z) = H_0 \prod_{i=1}^{2} H_i(z) \quad \text{where} \quad H_i(z) = \frac{(z - z_i)(z - z_i^*)}{(z - p_i)(z - p_i^*)}$$

with

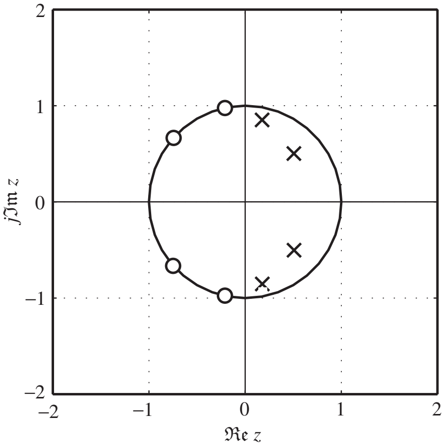
$$\begin{aligned} z_1, z_1^* &= -0.7470 \pm j0.6648 \\ z_2, z_2^* &= -0.2094 \pm j0.9778 \\ p_1, p_1^* &= \phantom{-}0.5078 \pm j0.5036 \\ p_2, p_2^* &= \phantom{-}0.1774 \pm j0.9233 \end{aligned}$$

and

$$H_0 = 6.351486E - 02$$

# Lowpass Filter *Cont'd*

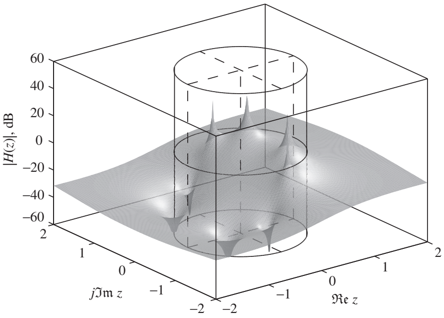
- Zero-pole plot:




(*a*)

# Lowpass Filter *Cont'd*

- Plot of $|H(z)|$ (in dB) versus $z = \text{Re } z + j\text{Im } z$:




(*b*)

The *dimples and spikes* are the *zeros and poles*, respectively.

# Lowpass Filter *Cont'd*

- The amplitude response can be obtained as

$$M(\omega) = |H_0| \prod_{i=1}^{2} \left| H_i(e^{j\omega T}) \right| = |H_0| \prod_{i=1}^{2} M_i(\omega) \quad \text{where}$$

$$\begin{aligned}
M_i(\omega) &= \left| H_i(e^{j\omega T}) \right| = \left| \frac{a_{0i} + a_{1i} e^{j\omega T} + e^{j2\omega T}}{b_{0i} + b_{1i} e^{j\omega T} + e^{j2\omega T}} \right| \\
&= \left| \frac{(a_{0i} + a_{1i} \cos \omega T + \cos 2\omega T) + j(a_{1i} \sin \omega T + \sin 2\omega T)}{(b_{0i} + b_{1i} \cos \omega T + \cos 2\omega T) + j(b_{1i} \sin \omega T + \sin 2\omega T)} \right| \\
&= \left[ \frac{(a_{0i} + a_{1i} \cos \omega T + \cos 2\omega T)^2 + (a_{1i} \sin \omega T + \sin 2\omega T)^2}{(b_{0i} + b_{1i} \cos \omega T + \cos 2\omega T)^2 + (b_{1i} \sin \omega T + \sin 2\omega T)^2} \right]^{\frac{1}{2}} \\
&= \left[ \frac{1 + a_{0i}^2 + a_{1i}^2 + 2(1 + a_{0i}) a_{1i} \cos \omega T + 2 a_{0i} \cos 2\omega T}{1 + b_{0i}^2 + b_{1i}^2 + 2(1 + b_{0i}) b_{1i} \cos \omega T + 2 b_{0i} \cos 2\omega T} \right]^{\frac{1}{2}}
\end{aligned}$$

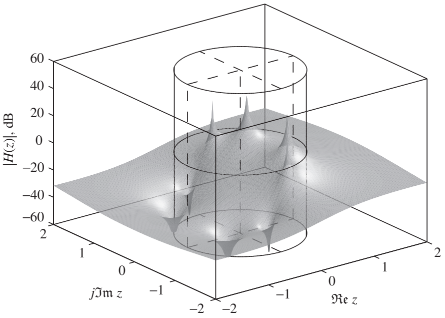
- Since $z = e^{j\omega nT}$ represents a circle of unit radius in the $z$ plane, the amplitude response

$$M(\omega) = |H(e^{j\omega nT})|$$

can be represented geometrically by the intersection between the surface $|H(z)|$ and a cylinder of unit radius perpendicular to the $z$ plane.

# Lowpass Filter *Cont'd*

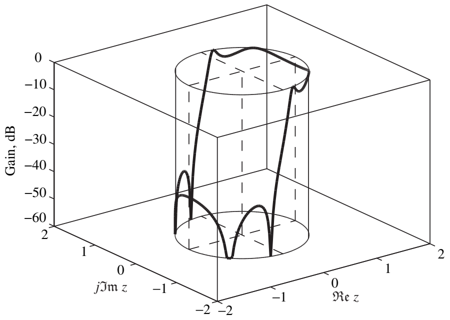
- Plot of $|H(z)|$ (in dB) versus $z = \text{Re } z + j\text{Im } z$:




(*b*)

The *intersection* between the surface $|H(z)|$ and the cylinder is the *amplitude response*.

- Plot of $|H(z)|$ (in dB) versus $z = e^{j\omega T}$:

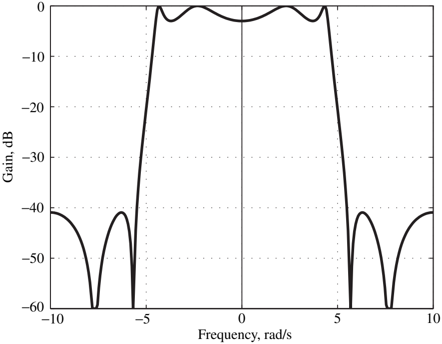



(*d*)

The *intersection* between surface $|H(z)|$ and the cylinder, i.e., the solid curve, is the *amplitude response*.

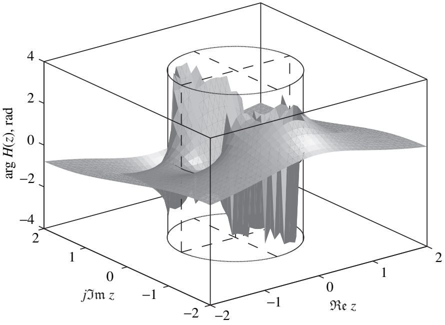
# Lowpass Filter *Cont'd*

- Slicing the cylinder along the vertical line $z = -1$ and flattening it out will reveal the amplitude response, i.e., $M(\omega)$ versus $\omega$, as a two-dimensional plot:




(*f*)

- Plot of arg $H(z)$ (in rad) versus $z = \text{Re } z + j\text{Im } z$:




(c)

# Lowpass Filter *Cont'd*

- The phase response can be obtained as

$$\theta(\omega) = \arg H_0 + \sum_{i=1}^{2} \arg H_i(e^{j\omega T}) = \sum_{i=1}^{2} \theta_i(\omega) \quad \text{where}$$

$$\begin{aligned}
\theta_i(\omega) &= \arg H_i(e^{j\omega T}) \\
&= \arg \frac{a_{0i} + a_{1i}e^{j\omega T} + e^{j2\omega T}}{b_{0i} + b_{1i}e^{j\omega T} + e^{j2\omega T}} \\
&= \arg \frac{(a_{0i} + a_{1i}\cos\omega T + \cos 2\omega T) + j(a_{1i}\sin\omega T + \sin 2\omega T)}{(b_{0i} + b_{1i}\cos\omega T + \cos 2\omega T) + j(b_{1i}\sin\omega T + \sin 2\omega T)} \\
&= \tan^{-1} \frac{a_{1i}\sin\omega T + \sin 2\omega T}{a_{0i} + a_{1i}\cos\omega T + \cos 2\omega T} \\
&\quad - \tan^{-1} \frac{b_{1i}\sin\omega T + \sin 2\omega T}{b_{0i} + b_{1i}\cos\omega T + \cos 2\omega T}
\end{aligned}$$

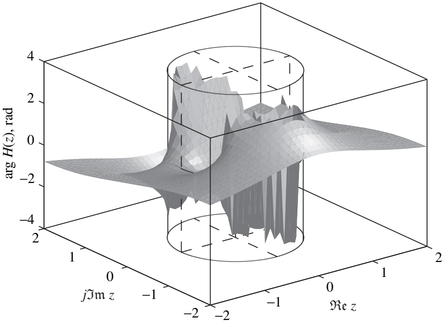
(See textbook for details.)

- Since $z = e^{j\omega T}$ represents a circle of unit radius in the $z$ plane, the phase response

$$\theta(\omega) = \arg H(e^{j\omega T}) = \tan^{-1} \frac{\text{Im } H(e^{j\omega T})}{\text{Re } H(e^{j\omega T})}$$

can be represented geometrically by the intersection between the surface $\arg H(z)$ and a cylinder of unit radius perpendicular to the $z$ plane.

- Plot of $\arg H(z)$ (in rad) versus $z = \text{Re } z + j\text{Im } z$:

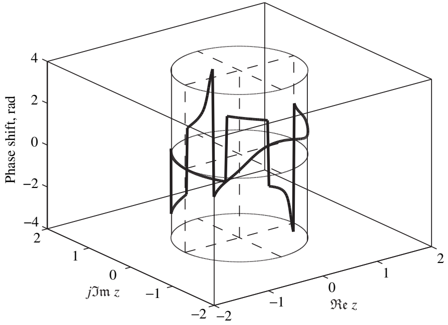



(c)

The *intersection* between surface $\arg H(z)$ and the cylinder is the *phase response*.

# Lowpass Filter *Cont'd*

- Plot of $\arg H(z)$ (in rad) versus $z = e^{j\omega T}$:

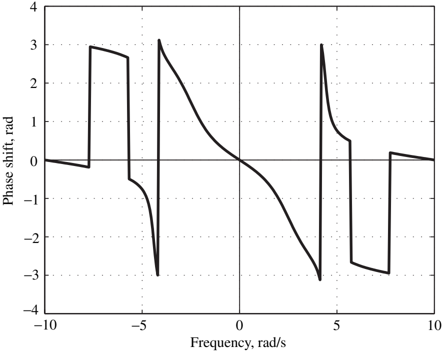



(*e*)

The *intersection* between surface $\arg H(z)$ and the cylinder, i.e., the solid curve, is the *phase response*.

# Lowpass Filter *Cont'd*

- Slicing the cylinder along the vertical line $z = -1$ and flattening it out will reveal the phase response, i.e., $\theta(\omega)$ versus $\omega$, as a two-dimensional plot:




(*g*)

# Pitfall

- The phase response shown in the previous slide is actually the phase response that would be computed by using MATLAB's function atan2(y,x) but *it is not correct*!

  The abrupt jumps of $2\pi$ should not be present.

# Pitfall

- The phase response shown in the previous slide is actually the phase response that would be computed by using MATLAB's function atan2(y,x) but *it is not correct*!

  The abrupt jumps of $2\pi$ should not be present.

- This problem has to do with the fact that

$$\theta = \tan^{-1} \frac{x}{y}$$

  is a multivalued function, and MATLAB's function atan2(y,x) would give a value for $\theta$ in the range $-2\pi \leq \theta \leq 2\pi$.

  Computers in general would give a value in the range $-\pi \leq \theta \leq \pi$.

# Pitfall

- The phase response shown in the previous slide is actually the phase response that would be computed by using MATLAB's function atan2(y,x) but *it is not correct*!

  The abrupt jumps of $2\pi$ should not be present.

- This problem has to do with the fact that

$$\theta = \tan^{-1} \frac{x}{y}$$

  is a multivalued function, and MATLAB's function atan2(y,x) would give a value for $\theta$ in the range $-2\pi \leq \theta \leq 2\pi$.

  Computers in general would give a value in the range $-\pi \leq \theta \leq \pi$.

- The problem can be corrected by noting that the phase response is a continuous function of $\omega$.

- For example, if function atan2(y,x) gives a value of $-179$ followed by a value of $+179^\circ$ then, assuming a continuous phase response, an error of $+360^\circ$ has been committed and $360^\circ$ should be subtracted from $+179^\circ$ to give the correct value of $-181^\circ$.

# Pitfall *Cont'd*

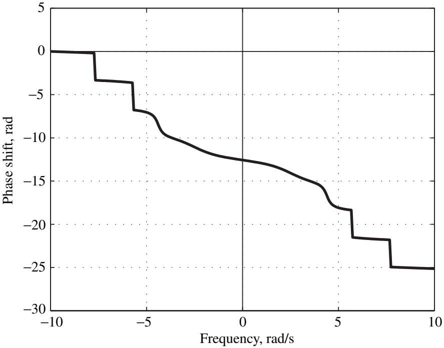
- For example, if function atan2(y,x) gives a value of $-179$ followed by a value of $+179^\circ$ then, assuming a continuous phase response, an error of $+360^\circ$ has been committed and $360^\circ$ should be subtracted from $+179^\circ$ to give the correct value of $-181^\circ$.
- Similarly, if function atan2(y,x) gives a value of $+179$ followed by a value of $-179^\circ$, then an error of $-360^\circ$ has been committed and $360^\circ$ should be added to $-179^\circ$ to give the correct value $+181^\circ$.

# Pitfall *Cont'd*

- For example, if function atan2(y,x) gives a value of $-179$ followed by a value of $+179^\circ$ then, assuming a continuous phase response, an error of $+360^\circ$ has been committed and $360^\circ$ should be subtracted from $+179^\circ$ to give the correct value of $-181^\circ$.
- Similarly, if function atan2(y,x) gives a value of $+179$ followed by a value of $-179^\circ$, then an error of $-360^\circ$ has been committed and $360^\circ$ should be added to $-179^\circ$ to give the correct value $+181^\circ$.
- Alternatively, the correct value of the phase response can be obtained by using function unwrap(p) of MATLAB, which will perform the necessary corrections.

# Lowpass Filter *Cont'd*

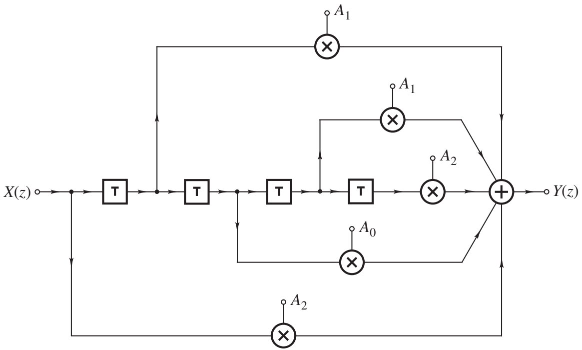
- Unwrapped phase response:




(*i*)

# Example – Nonrecursive Lowpass Filter

The figure shows a nonrecursive filter:




$$A_0 = 0.3352, \quad A_1 = 0.2540, \quad A_2 = 0.0784$$

# Example *Cont'd*

($a$) Construct the zero-pole plot.

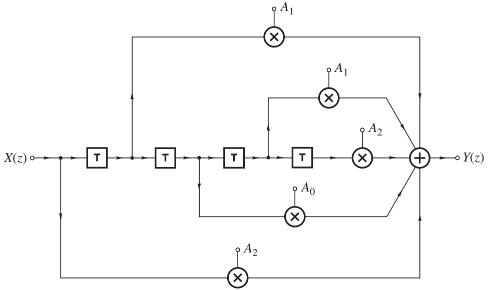
($b$) Plot the surface $|H(z)|$ as a function of $z = \text{Re } z + j\text{Im } z$.

($c$) Obtain expressions for the amplitude and phase responses.

($d1$) Plot the amplitude and phase responses in terms of 3-D plots.

($d2$) Plot the amplitude and phase responses in terms of 2-D plots.

## Solution




Transfer function:

$$
\begin{aligned}
H(z) &= A_2 + A_1 z^{-1} + A_0 z^{-2} + A_1 z^{-3} + A_2 z^{-4} \\
&= \frac{A_2 z^2 + A_1 z + A_0 + A_1 z^{-1} + A_2 z^{-2}}{z^2} \\
&= \frac{A_2 z^4 + A_1 z^3 + A_0 z^2 + A_1 z + A_2}{z^4}
\end{aligned}
$$

# Example *Cont'd*

. . .

$$\begin{aligned} H(z) &= A_2 + A_1 z^{-1} + A_0 z^{-2} + A_1 z^{-3} + A_2 z^{-4} \\ &= \frac{A_2 z^2 + A_1 z + A_0 + A_1 z^{-1} + A_2 z^{-2}}{z^2} \\ &= \frac{A_2 z^4 + A_1 z^3 + A_0 z^2 + A_1 z + A_2}{z^4} \end{aligned}$$

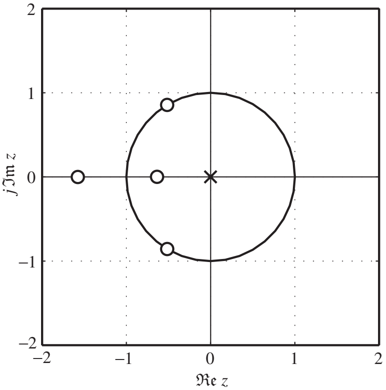
The zeros can be readily found by using D-Filter or MATLAB as

$$z_1 = -1.5756 \quad z_2 = -0.6347 \quad z_3, z_4 = -0.5148 \pm j0.8573$$

There is a 4th-order pole at the origin.

# Example *Cont'd*

Zero-pole plot:



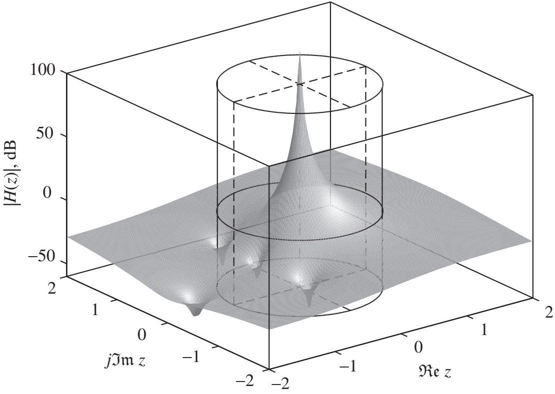

(*a*)

$$z_1 = -1.5756 \quad z_2 = -0.6347 \quad z_3, z_4 = -0.5148 \pm j0.8573$$

$$p_1 = p_2 = p_3 = p_4 = 0$$

# Example *Cont'd*

$|H(z)|$ versus $z = \text{Re } z + j\text{Im } z$:




(*b*)

Dimples represent zeros, the huge spike represents the 4th-order pole at the origin.

# Example *Cont'd*

Since

$$\begin{aligned} H(z) &= A_2 + A_1 z^{-1} + A_0 z^{-2} + A_1 z^{-3} + A_2 z^{-4} \\ &= \frac{A_2 z^2 + A_1 z + A_0 + A_1 z^{-1} + A_2 z^{-2}}{z^2} \qquad \text{(A)} \\ &= \frac{A_2 z^4 + A_1 z^3 + A_0 z^2 + A_1 z + A_2}{z^4} \end{aligned}$$

Eq. (A) gives the frequency response as

$$\begin{aligned} H(e^{j\omega T}) &= \frac{A_2(e^{j2\omega T} + e^{-j2\omega T}) + A_1(e^{j\omega T} + e^{-j\omega T}) + A_0}{e^{j2\omega T}} \\ &= \frac{2A_2 \cos 2\omega T + 2A_1 \cos \omega T + A_0}{e^{j2\omega T}} \end{aligned}$$

# Example *Cont'd*

...

$$H(e^{j\omega T}) = \frac{2A_2 \cos 2\omega T + 2A_1 \cos \omega T + A_0}{e^{j2\omega T}}$$

Therefore, the amplitude and phase responses are given by

$$M(\omega) = |2A_2 \cos 2\omega T + 2A_1 \cos \omega T + A_0| \quad \blacksquare$$

and

$$\theta(\omega) = \theta_N - 2\omega T \quad \blacksquare$$

respectively, where

$$\theta_N = \begin{cases} 0 & \text{if } 2A_2 \cos 2\omega T + 2A_1 \cos \omega T + A_0 \geq 0 \\ \pi & \text{otherwise} \end{cases}$$

...

$$H(e^{j\omega T}) = \frac{2A_2 \cos 2\omega T + 2A_1 \cos \omega T + A_0}{e^{j2\omega T}}$$

Therefore, the amplitude and phase responses are given by

$$M(\omega) = |2A_2 \cos 2\omega T + 2A_1 \cos \omega T + A_0| \quad \blacksquare$$

and

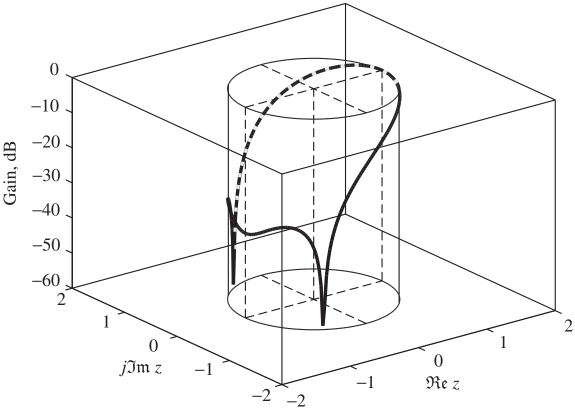
$$\theta(\omega) = \theta_N - 2\omega T \quad \blacksquare$$

respectively, where

$$\theta_N = \begin{cases} 0 & \text{if } 2A_2 \cos 2\omega T + 2A_1 \cos \omega T + A_0 \geq 0 \\ \pi & \text{otherwise} \end{cases}$$

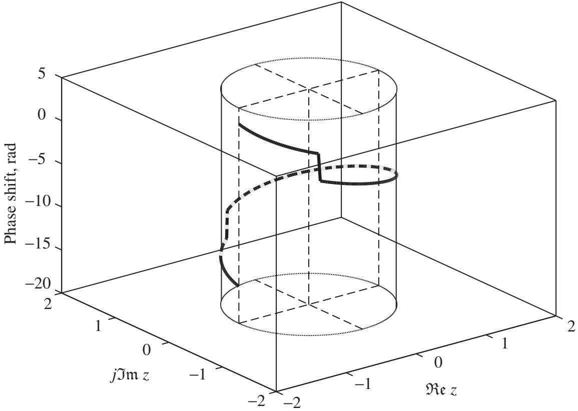
*Note:* The phase response is usually a linear function of $\omega$ in nonrecursive filters (see Chap. 9).

3-D plot of amplitude response, i.e., arg $H(e^{j\omega T})$ versus $z = e^{j\omega T}$:




(*c*)

3-D plot of phase response, i.e., $\arg H(e^{j\omega T})$ versus $z = e^{j\omega T}$:

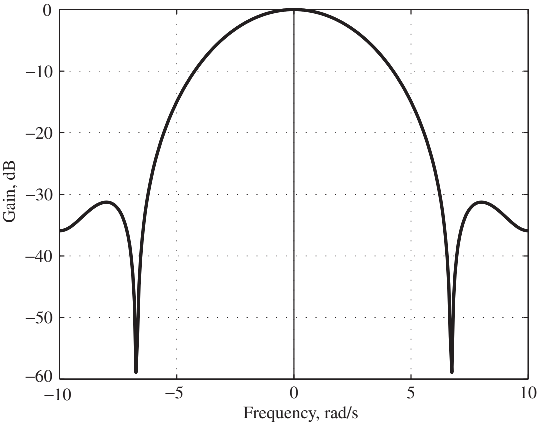



(*d*)

*Note:* The phase angle has been unwrapped.

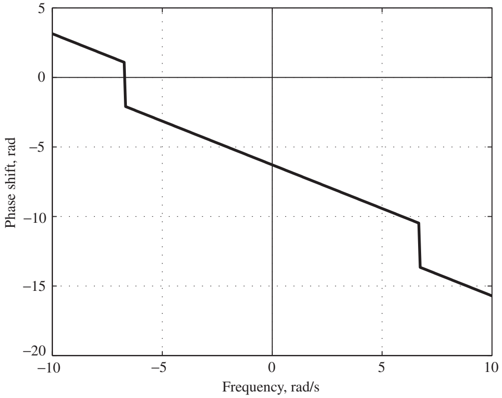
# Example *Cont'd*

2-D plot of amplitude response, i.e., $M(\omega) = |H(e^{j\omega T})|$ versus $\omega$:




(*e*)

2-D plot of phase response, i.e., $\arg H(e^{j\omega T})$ versus $\omega$:




(*f*)

*Note:* The discontinuities are genuine: they are caused by zeros on the unit circle.

# Example – Recursive Bandpass Filter

A recursive digital filter is characterized by the transfer function

$$H(z) = H_0 \prod_{i=1}^{3} H_i(z)$$

where

$$H_i(z) = \frac{a_{0i} + a_{1i}z + z^2}{b_{0i} + b_{1i}z + z^2}$$

The sampling frequency is 20 rad/s.

Transfer-Function Coefficients

| $i$ | $a_{0i}$ | $a_{1i}$ | $b_{0i}$ | $b_{1i}$ |
|---|---|---|---|---|
| 1 | −1.0 | 0.0 | 8.131800E−1 | 7.870090E−8 |
| 2 | 1.0 | −1.275258 | 9.211099E−1 | 5.484026E−1 |
| 3 | 1.0 | 1.275258 | 9.211097E−1 | −5.484024E−1 |
| | $H_0 = 1.763161E-2$ | | | |

# Example *Cont'd*

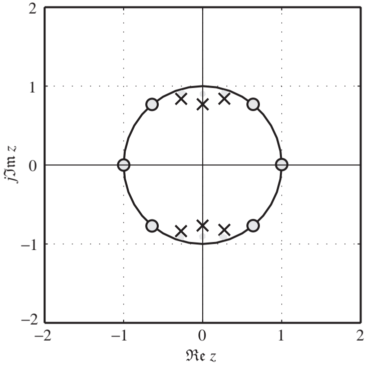
(*a*) Construct the zero-pole plot of the filter.

(*b*) Plot the surface $|H(z)|$ as a function of $z = \text{Re } z + j\text{Im } z$.

(*c*) Obtain expressions for the amplitude and phase responses.

(*d*) Plot the amplitude and phase responses first in terms of 3-D plots and then in terms of 2-D plots.

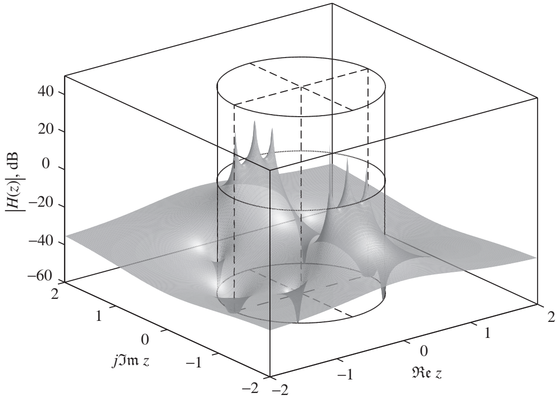
## Solution




(a)

$$z_1, z_2 = \pm 1 \quad z_3, z_4 = 0.638 \pm j0.770 \quad z_5, z_6 = -0.638 \pm j0.770$$
$$p_1, p_2 = \pm j0.902 \quad p_3, p_4 = 0.274 \pm j0.770 \quad p_5, p_6 = -0.274 \pm j0.770$$

$|H(z)|$ versus $z = \text{Re } z + j\text{Im } z$:

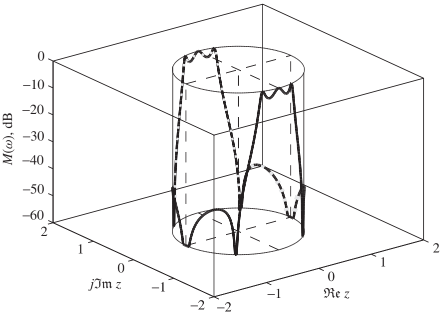



(*b*)

Dimples represent zeros, the huge spike represents the 4th-order pole at the origin.

# Example *Cont'd*

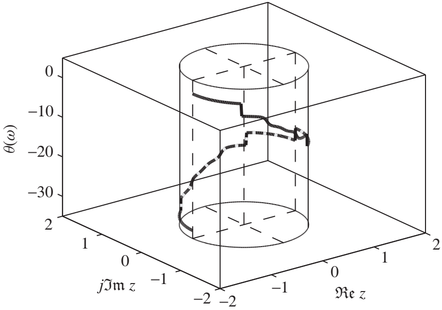
Plot of $|H(z)|$ (in dB) versus $z = e^{j\omega T}$:



(c)

The *intersection* between surface $|H(z)|$ and the cylinder, i.e., the solid curve, is the *amplitude response*.

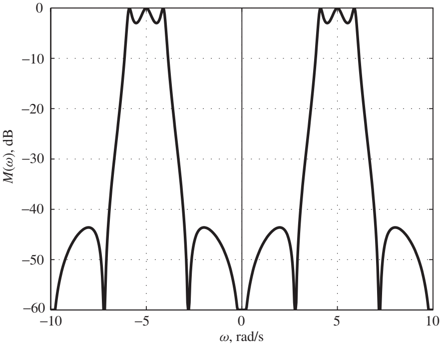
Plot of $\arg H(z)$ (in rad) versus $z = e^{j\omega T}$ with the phase angle unwrapped:




(*d*)

# Example *Cont'd*

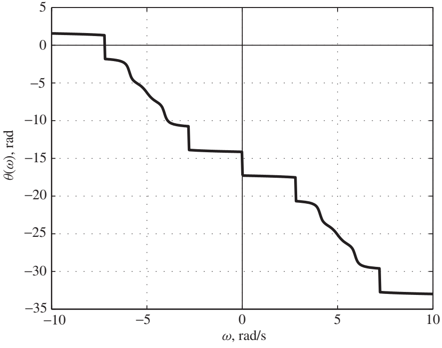
Slicing the cylinder along the vertical line $z = -1$ and flattening it out will reveal the amplitude response, i.e., $M(\omega)$ versus $\omega$, as a two-dimensional plot:




(*e*)

# Example *Cont'd*

Unwrapped phase response:



(*f*)

*Note:* The discontinuities shown are genuine. They are caused by the zeros on the unit circle.

*This slide concludes the presentation.*
*Thank you for your attention.*