Chapter 5
THE APPLICATION OF THE Z TRANSFORM
5.6 Transfer Functions for Digital Filters
5.7 Amplitude and Delay Distortion

Victoria, BC, Canada
Email: aantoniou@ieee.org

July 14, 2018

Introduction

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- ⇛ In this presentation, we examine some of the basic types of transfer functions that characterize some typical first- and second-order filter types known as *biquads*.
- ⇛ Biquads are often used as basic digital-filter blocks to construct high-order filters.

First-Order Transfer Functions

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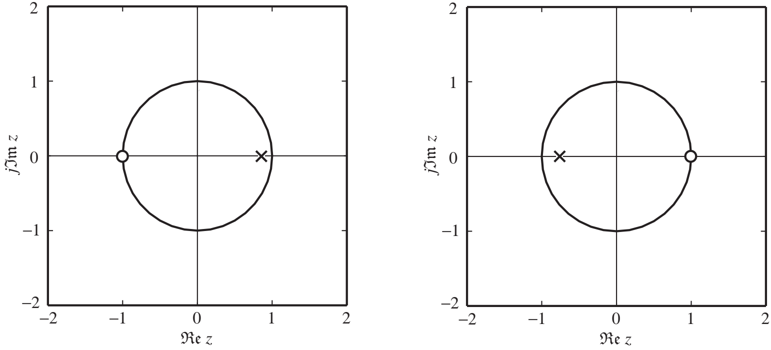
- The zero can be anywhere on the real axis of the z plane.

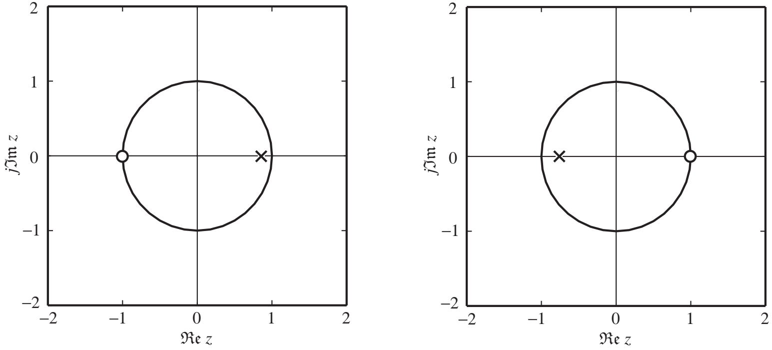
⇒ If the pole is close to point (1, 0) and the zero is close to or at point (−1, 0), then we have a *lowpass* filter.



First-Order Transfer Functions *Cont'd*

⇛ If the pole is close to point (1, 0) and the zero is close to or at point (−1, 0), then we have a *lowpass* filter.

⇛ If the zero and pole positions are interchanged, then we get a *highpass* filter.



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$$H(z) = \frac{p_0 z - 1}{z - p_0} = p_0 \frac{z - 1/p_0}{z - p_0}$$

where the zero is the reciprocal of the pole.

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where the zero is the reciprocal of the pole.

⇛ The frequency response of a system characterized by $H(z)$ is given by

$$H(e^{j\omega T}) = \frac{p_0 e^{j\omega T} - 1}{e^{j\omega T} - p_0} = \frac{p_0 \cos \omega T + j p_0 \sin \omega T - 1}{\cos \omega T + j \sin \omega T - p_0}$$

First-Order Transfer Functions *Cont'd*

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⇛ The amplitude and phase responses are given by

$$M(\omega) = \left| \frac{p_0 \cos\omega T - 1 + jp_0 \sin\omega T}{\cos\omega T - p_0 + j\sin\omega T} \right|$$

$$= \left[\frac{(p_0 \cos\omega T - 1)^2 + (p_0 \sin\omega T)^2}{(\cos\omega T - p_0)^2 + (\sin\omega T)^2} \right]^{\frac{1}{2}} = 1$$

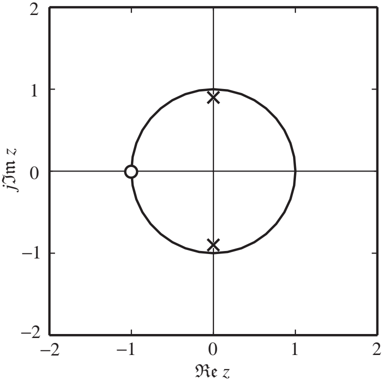
and

$$\theta(\omega) = \tan^{-1} \frac{p_0 \sin\omega T}{p_0 \cos\omega T - 1} - \tan^{-1} \frac{\sin\omega T}{\cos\omega T - p_0}$$

respectively.

Second-Order Lowpass Biquad

⇛ A *lowpass* second-order transfer function can be constructed by placing a complex-conjugate pair of poles anywhere inside the unit circle and a pair of zeros at the Nyquist point:



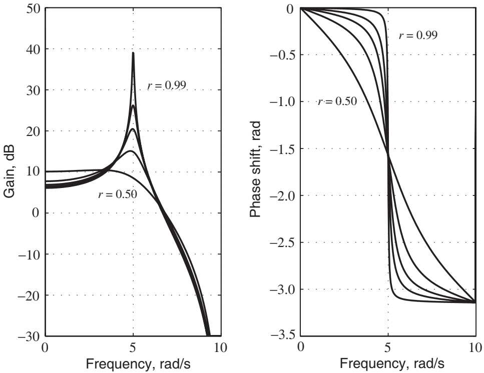
Second-Order Lowpass Biquad *Cont'd*

⇛ The transfer function of the lowpass biquad assumes the form:

$$H_{LP}(z) = \frac{(z+1)^2}{(z - re^{j\phi})(z - re^{-j\phi})} = \frac{z^2 + 2z + 1}{z^2 - 2r(\cos\phi)z + r^2}$$

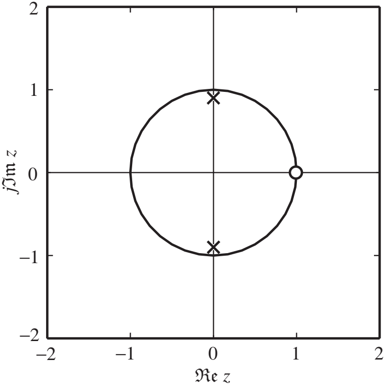
where $0 < r < 1$.

⇛ As the poles move closer to the unit circle, the amplitude response develops a peak at frequency $\omega = \phi/T$ while the slope of the phase response tends to become steeper and steeper at that frequency.



Second-Order Highpass Biquad

⇛ A *highpass* second-order transfer function can be constructed by placing a complex-conjugate pair of poles anywhere inside the unit circle and a pair of zeros at point $(1, 0)$:



Second-Order Highpass Biquad *Cont'd*

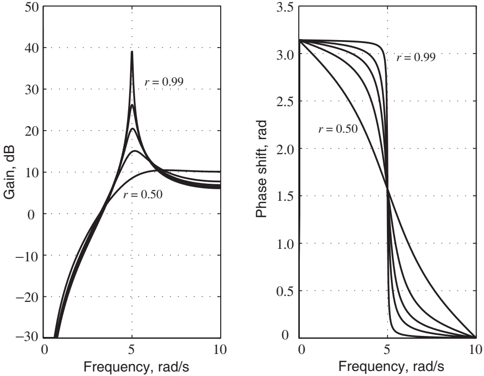
⇛ The transfer function of the highpass biquad assumes the form:

$$H_{HP}(z) = \frac{(z-1)^2}{z^2 - 2r(\cos\phi)z + r^2} = \frac{(z^2 - 2z + 1)}{z^2 - 2r(\cos\phi)z + r^2}$$

where $0 < r < 1$.

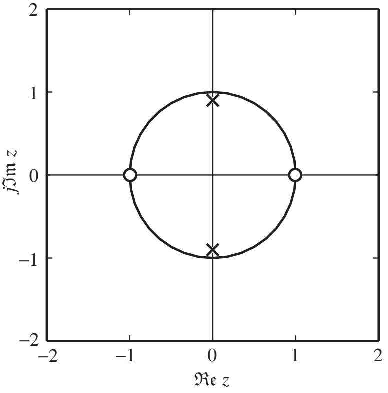
Second-Order Highpass Biquad *Cont'd*

⇛ As the poles move closer to the unit circle, the amplitude response develops a peak at frequency $\omega = \phi/T$ while the slope of the phase response tends to become steeper and steeper at that frequency.



Second-Order Bandpass Biquad

⇛ A *bandpass* second-order transfer function can be constructed by placing a complex-conjugate pair of poles anywhere inside the unit circle, zeros at points (-1, 0) and (1, 0):



Second-Order Bandpass Biquad *Cont'd*

⇛ The transfer function of the bandpass biquad assumes the form:

$$H_{BP}(z) = \frac{(z+1)(z-1)}{z^2 - 2r(\cos\phi)z + r^2}$$

where $0 < r < 1$.

Second-Order Bandpass Biquad *Cont'd*

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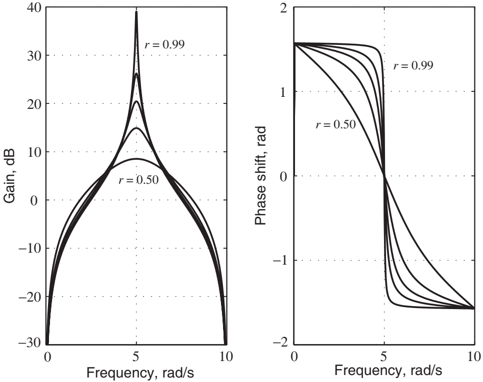
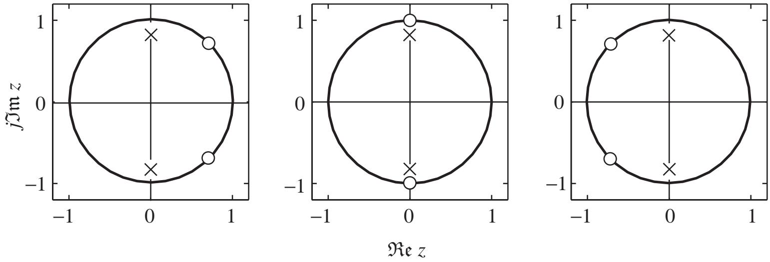


Second-Order Notch Biquad

⇛ A *notch* second-order transfer function can be constructed by placing a complex-conjugate pair of poles anywhere inside the unit circle, and a complex-conjugate pair of zeros on the unit circle.

There are three possibilities:



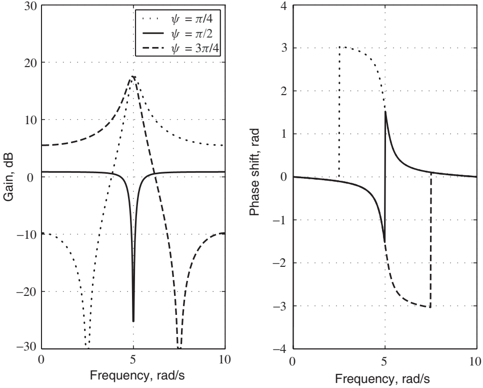
Second-Order Notch Biquad *Cont'd*

⇛ The transfer function of the bandpass biquad assumes the form:

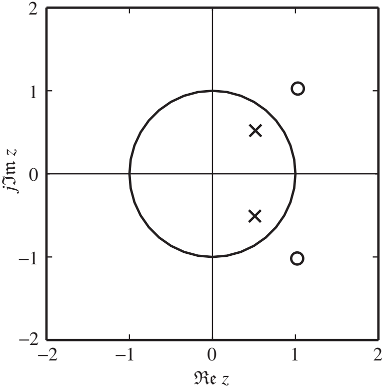
$$H_N(z) = \frac{z^2 - 2(\cos\psi)z + 1}{z^2 - 2r(\cos\phi)z + r^2}$$

where $0 < r < 1$.

⇛ If $\psi = \pi/4$, $\psi = \pi/2$, or $\psi = 3\pi/4$, the notch filter behaves as a highpass, bandstop, or lowpass filter.



⇛ An *allpass* second-order transfer function can be constructed by placing a complex-conjugate pair of poles anywhere inside the unit circle and a complex-conjugate pair of zeros that are the reciprocals of the poles outside the unit circle.



⇛ The transfer function of the bandpass biquad assumes the form:

$$H_{AP}(z) = \frac{r^2z^2 - 2r(\cos\phi)z + 1}{z^2 - 2r(\cos\phi)z + r^2}$$

where $0 < r < 1$.

Second-Order Allpass Biquad *Cont'd*

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⇛ We note that the numerator coefficients are the same as the denominator coefficients but in the reverse order.

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⇛ The above is a general property, that is, an arbitrary transfer function with the above coefficient symmetry is an allpass transfer function independently of the order.

$$M_{AP}(\omega)$$

$$= |H_{AP}(e^{j\omega T})| = \left[H_{AP}(e^{j\omega T}) \cdot H_{AP}^*(e^{j\omega T})\right]^{\frac{1}{2}}$$

$$= \left[H_{AP}(e^{j\omega T}) \cdot H_{AP}(e^{-j\omega T})\right]^{\frac{1}{2}}$$

$$= \left\{\left[H_{AP}(z) \cdot H_{AP}(z^{-1})\right]_{z=e^{j\omega T}}\right\}^{\frac{1}{2}}$$

$$= \left\{\left[\frac{r^2z^2 + 2r(\cos\phi)z + 1}{z^2 + 2r(\cos\phi)z + r^2} \cdot \frac{r^2z^{-2} + 2r(\cos\phi)z^{-1} + 1}{z^{-2} + 2r(\cos\phi)z^{-1} + r^2}\right]_{z=e^{j\omega T}}\right\}^{\frac{1}{2}}$$

$$= \left\{\left[\frac{r^2z^2 + 2r(\cos\phi)z + 1}{z^2 + 2r(\cos\phi)z + r^2} \cdot \frac{r^2 + 2r(\cos\phi)z + z^2}{1 + 2r(\cos\phi)z + z^2r^2}\right]_{z=e^{j\omega T}}\right\}^{\frac{1}{2}} = 1$$

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- ⇛ Corresponding high-order filters can be constructed by connecting several biquads in cascade or in parallel.
- ⇛ Methods for obtaining transfer functions that will yield specified frequency responses will be explored in later chapters.

Amplitude and Delay Distortion

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- Amplitude distortion
- Delay (or phase) distortion

⇒ Consider an application where a digital filter characterized by a transfer function $H(z)$ is to be used to select a specific signal $x_k(nT)$ from a sum of signals

$$x(nT) = \sum_{i=1}^{m} x_i(nT)$$

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⇛ Let the amplitude and phase responses of the filter be $M(\omega)$ and $\theta(\omega)$, respectively.

⇛ Two parameters associated with the phase response are the *absolute delay* $\tau_a(\omega)$ and the *group delay* $\tau_g(\omega)$ which are defined as

$$\tau_a(\omega) = -\frac{\theta(\omega)}{\omega} \quad \text{and} \quad \tau_g(\omega) = -\frac{d\theta(\omega)}{d\omega}$$

Amplitude and Delay Distortion *Cont'd*

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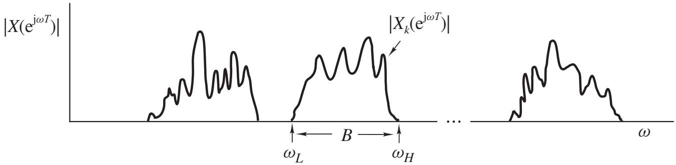
- As functions of frequency, $\tau_a(\omega)$ and $\tau_g(\omega)$ are known as the *absolute-delay and group-delay characteristics*.

⇛ Now assume that the amplitude spectrum of signal $x_k(nT)$ is concentrated in frequency band B given by

$$B = \{\omega : \omega_L \leq \omega \leq \omega_H\}$$

as shown.



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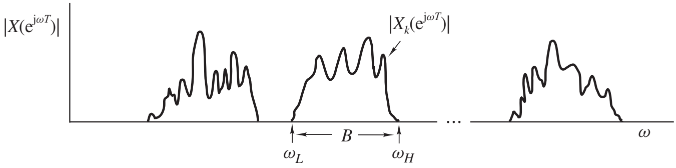
⇛ Also assume that the filter has amplitude and phase responses

$$M(\omega) = \begin{cases} G_0 & \text{for } \omega \in B \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \theta(\omega) = -\tau_g \omega + \theta_0 \quad \text{for} \quad \omega \in B$$

respectively, where G_0 and τ_g are constants.



Amplitude and Delay Distortion *Cont'd*

⇛ The z transform of the output of the filter is given by

$$Y(z) = H(z)X(z) = H(z)\sum_{i=1}^{m} X_i(z) = \sum_{i=1}^{m} H(z)X_i(z)$$

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⇛ Thus the frequency spectrum of the output signal is obtained as

$$\begin{aligned} Y(e^{j\omega T}) &= \sum_{i=1}^{m} H(e^{j\omega T})X_i(e^{j\omega T}) \\ &= \sum_{i=1}^{m} M(\omega)e^{j\theta(\omega)}X_i(e^{j\omega T}) \end{aligned}$$

Amplitude and Delay Distortion *Cont'd*

...

$$Y(e^{j\omega T}) = \sum_{i=1}^{m} M(\omega)e^{j\theta(\omega)}X_i(e^{j\omega T})$$

⇛ We have assumed that

$$M(\omega) = \begin{cases} G_0 & \text{for } \omega \in B \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \theta(\omega) = -\tau_g\omega + \theta_0 \quad \text{for} \quad \omega \in B$$

and hence we get

$$Y(e^{j\omega T}) = G_0 e^{-j\omega\tau_g + j\theta_0} X_k(e^{j\omega T})$$

since all signal spectrums except $X_k(e^{j\omega T})$ will be multiplied by zero.

Amplitude and Delay Distortion *Cont'd*

...

$$Y(e^{j\omega T}) = G_0 e^{-j\omega\tau_g + j\theta_0} X_k(e^{j\omega T})$$

⇛ If we now let $\tau_g = mT$ where m is a constant, we can write

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$$y(nT) = G_0 e^{j\theta_0} x_k(nT - mT)$$

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⇛ In effect, if the *amplitude response* of the filter is *constant* with respect to frequency band B and zero elsewhere and its *phase response* is a *linear* function of ω, that is, the group delay is constant in frequency band B, then the output signal is a *delayed replica* of signal $x_k(nT)$ except that a constant multiplier $G_0 e^{j\theta_0}$ is introduced.

⇛ If the amplitude response of the system is not constant in frequency band B, then so-called *amplitude distortion* will be introduced since different frequency components of the signal will be amplified by different amounts.

Amplitude and Delay Distortion *Cont'd*

⇛ If the amplitude response of the system is not constant in frequency band B, then so-called *amplitude distortion* will be introduced since different frequency components of the signal will be amplified by different amounts.

⇛ If the group delay is not constant in band B, different frequency components will be delayed by different amounts, and *delay (or phase) distortion* will be introduced.

- Amplitude distortion can be quite objectionable in practice.

  Consequently, the amplitude response is required to be flat to within a prescribed tolerance in each frequency band that carries information.

- If the ultimate receiver of the signal is the human ear, e.g., when a speech or music signal is to be processed, delay distortion turns out to be quite tolerable.

- In other applications where images are involved, e.g., transmission of video signals, delay distortion can be as objectionable as amplitude distortion, and the delay characteristic is required to be fairly flat.

This slide concludes the presentation.
Thank you for your attention.