Chapter 9
DESIGN OF NONRECURSIVE (FIR) FILTERS
9.1 Introduction
9.2.1 Properties of Constant-Delay Nonrecursive Filters

Victoria, BC, Canada
Email: aantoniou@ieee.org

July 14, 2018

Introduction

- As shown in Sec. 5.7 of the textbook, a nonlinear phase response will introduce *delay distortion*.

Introduction

- As shown in Sec. 5.7 of the textbook, a nonlinear phase response will introduce *delay distortion*.
- In certain filtering applications, delay distortion is highly undesirable and in such applications digital filters are required that have a *linear-phase response*.

Introduction

- As shown in Sec. 5.7 of the textbook, a nonlinear phase response will introduce *delay distortion*.
- In certain filtering applications, delay distortion is highly undesirable and in such applications digital filters are required that have a *linear-phase response*.
- It turns out that linear-phase response can be easily achieved by designing the required filter as a nonrecursive filter.

Introduction

- As shown in Sec. 5.7 of the textbook, a nonlinear phase response will introduce *delay distortion*.
- In certain filtering applications, delay distortion is highly undesirable and in such applications digital filters are required that have a *linear-phase response*.
- It turns out that linear-phase response can be easily achieved by designing the required filter as a nonrecursive filter.
- A linear-phase response is obtained by simply ensuring that the impulse response satisfies certain symmetry conditions.

Introduction

- As shown in Sec. 5.7 of the textbook, a nonlinear phase response will introduce *delay distortion*.
- In certain filtering applications, delay distortion is highly undesirable and in such applications digital filters are required that have a *linear-phase response*.
- It turns out that linear-phase response can be easily achieved by designing the required filter as a nonrecursive filter.
- A linear-phase response is obtained by simply ensuring that the impulse response satisfies certain symmetry conditions.
- In this presentation, some basic properties of linear-phase nonrecursive filters are examined.

Constant Delay in Nonrecursive Filters

- A causal nonrecursive filter can be represented by the transfer function

$$H(z) = \sum_{n=0}^{N-1} h(nT) z^{-n}$$

Constant Delay in Nonrecursive Filters

- A causal nonrecursive filter can be represented by the transfer function

$$H(z) = \sum_{n=0}^{N-1} h(nT) z^{-n}$$

- Its frequency response is given by

$$H(e^{j\omega T}) = M(\omega)e^{j\theta(\omega)} = \sum_{n=0}^{N-1} h(nT)e^{-j\omega nT}$$

where

$$M(\omega) = |H(e^{j\omega T})| \quad \text{and} \quad \theta(\omega) = \arg H(e^{j\omega T})$$

...

$$\theta(\omega) = \arg H(e^{j\omega T})$$

- The absolute delay, which is also known as the *phase delay*, and the *group delay* of a filter are given by

$$\tau_p = -\frac{\theta(\omega)}{\omega} \quad \text{and} \quad \tau_g = -\frac{d\theta(\omega)}{d\omega}$$

...

$$\theta(\omega) = \arg H(e^{j\omega T})$$

- The absolute delay, which is also known as the *phase delay*, and the *group delay* of a filter are given by

$$\tau_p = -\frac{\theta(\omega)}{\omega} \quad \text{and} \quad \tau_g = -\frac{d\theta(\omega)}{d\omega}$$

- If both the phase and group delays are assumed to be constant, then the phase response must be linear, i.e.,

$$\theta(\omega) = -\tau\omega = \tan^{-1} \frac{-\sum_{n=0}^{N-1} h(nT)\sin\omega nT}{\sum_{n=0}^{N-1} h(nT)\cos\omega nT}$$

where τ is a constant.

Constant Delay in Nonrecursive Filters *Cont'd*

...

$$\theta(\omega) = -\tau\omega = \tan^{-1} \frac{-\sum_{n=0}^{N-1} h(nT) \sin \omega nT}{\sum_{n=0}^{N-1} h(nT) \cos \omega nT}$$

- Hence

$$\tan \tau\omega = \frac{\sum_{n=0}^{N-1} h(nT) \sin \omega nT}{\sum_{n=0}^{N-1} h(nT) \cos \omega nT}$$

or

$$\sum_{n=0}^{N-1} h(nT)(\cos \omega nT \sin \omega\tau - \sin \omega nT \cos \omega\tau) = 0$$

and so

$$\sum_{n=0}^{N-1} h(nT) \sin(\omega\tau - \omega nT) = 0$$

...

$$\sum_{n=0}^{N-1} h(nT)\sin(\omega\tau - \omega nT) = 0$$

- The solution of the above equation can be shown to be

$$\tau = \tfrac{1}{2}(N-1)T$$

$$h(nT) = h[(N-1-n)T] \quad \text{for} \quad 0 \le n \le N-1$$

...

$$\sum_{n=0}^{N-1} h(nT)\sin(\omega\tau - \omega nT) = 0$$

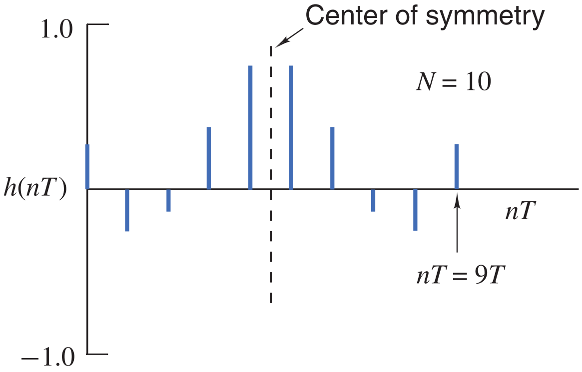
- The solution of the above equation can be shown to be

$$\tau = \tfrac{1}{2}(N-1)T$$

$$h(nT) = h[(N-1-n)T] \quad \text{for} \quad 0 \le n \le N-1$$

- Therefore, a nonrecursive filter can be designed to have *constant phase and group delays* over its entire baseband by simply ensuring that its impulse response is *symmetrical* about its center.

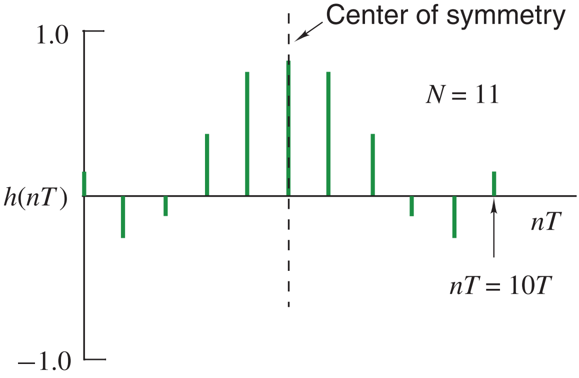
- For *even* N, the impulse response is symmetrical about the midpoint between samples $(N-2)/2$ and $N/2$ as shown:



(a)

Constant Delay in Nonrecursive Filters *Cont'd*

- For *odd* N, the impulse response is symmetrical about sample $(N-1)/2$ as shown:



(b)

- In most applications only the group delay needs to be constant in which case the phase response can have the form

$$\theta(\omega) = \theta_0 - \tau\omega$$

where θ_0 is a constant.

- In most applications only the group delay needs to be constant in which case the phase response can have the form

$$\theta(\omega) = \theta_0 - \tau\omega$$

where θ_0 is a constant.

- If we assume that $\theta_0 = \pm\pi/2$, a second class of constant-delay nonrecursive filters is obtained where

$$\begin{aligned} \tau &= \tfrac{1}{2}(N-1)T \\ h(nT) &= -h[(N-1-n)T] \end{aligned}$$

Constant Delay in Nonrecursive Filters *Cont'd*

- In most applications only the group delay needs to be constant in which case the phase response can have the form

$$\theta(\omega) = \theta_0 - \tau\omega$$

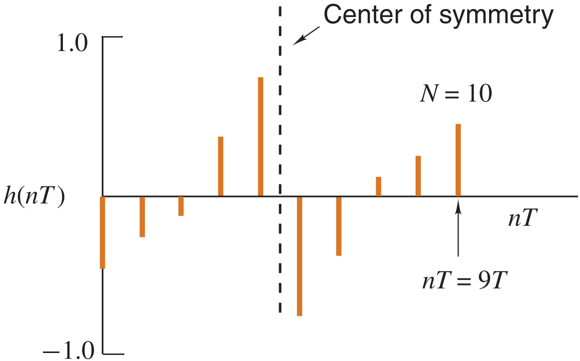
where θ_0 is a constant.

- If we assume that $\theta_0 = \pm\pi/2$, a second class of constant-delay nonrecursive filters is obtained where

$$\begin{aligned} \tau &= \tfrac{1}{2}(N-1)T \\ h(nT) &= -h[(N-1-n)T] \end{aligned}$$

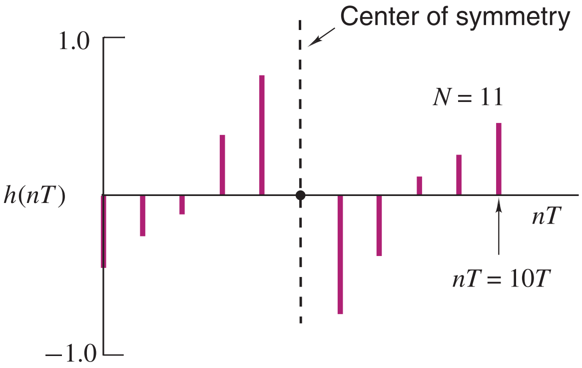
- In effect, a nonrecursive filter can be designed to have *constant group delay* over its entire baseband by simply ensuring that its impulse response is *antisymmetrical* about its center.

- For *even* N, the impulse response is antisymmetrical about the midpoint between samples $(N-2)/2$ and $N/2$ as shown:



(a)

- For *odd* N, the impulse response is antisymmetrical about sample $(N-1)/2$ as shown:



(b)

Frequency Response of Nonrecursive Filters

- The symmetries in the impulse response discussed lead to some simple expressions for the *frequency response* of nonrecursive filters.

Frequency Response of Nonrecursive Filters

- The symmetries in the impulse response discussed lead to some simple expressions for the *frequency response* of nonrecursive filters.
- For the case of a *symmetrical* impulse response and odd N,

$$H(e^{j\omega T}) = \sum_{n=0}^{(N-3)/2} h(nT)e^{-j\omega nT} + h\left[\frac{(N-1)T}{2}\right]e^{-j\omega(N-1)T/2} + \sum_{n=(N+1)/2}^{N-1} h(nT)e^{-j\omega nT} \qquad \text{(A)}$$

Frequency Response of Nonrecursive Filters

- The symmetries in the impulse response discussed lead to some simple expressions for the *frequency response* of nonrecursive filters.
- For the case of a *symmetrical* impulse response and odd N,

$$H(e^{j\omega T}) = \sum_{n=0}^{(N-3)/2} h(nT)e^{-j\omega nT} + h\left[\frac{(N-1)T}{2}\right]e^{-j\omega(N-1)T/2} + \sum_{n=(N+1)/2}^{N-1} h(nT)e^{-j\omega nT} \qquad \text{(A)}$$

- If we first let $N-1-n=m$ and then let $m=n$, we get

$$\sum_{n=(N+1)/2}^{N-1} h(nT)e^{-j\omega nT} = \sum_{n=(N+1)/2}^{N-1} h[(N-1-n)T]e^{-j\omega nT} = \sum_{n=0}^{(N-3)/2} h(nT)e^{-j\omega(N-1-n)T} \qquad \text{(B)}$$

- From Eqs. (A) and (B)

$$H(e^{j\omega T}) = e^{-j\omega(N-1)T/2}\left\{h\left[\frac{(N-1)T}{2}\right] + \sum_{n=0}^{(N-3)/2} 2h(nT)\cos\left[\omega\left(\frac{N-1}{2}-n\right)T\right]\right\}$$

and with $(N-1)/2 - n = k$, we have

$$H(e^{j\omega T}) = e^{-j\omega(N-1)T/2}\sum_{k=0}^{(N-1)/2} a_k \cos\omega kT$$

where $a_0 = h\left[\frac{(N-1)T}{2}\right]$ and $a_k = 2h\left[\left(\frac{N-1}{2}-k\right)T\right]$

$h(nT)$	N	$H(e^{j\omega T})$
Symmetrical	Odd	$e^{-j\omega(N-1)T/2}\sum_{k=0}^{(N-1)/2} a_k \cos \omega kT$
	Even	$e^{-j\omega(N-1)T/2}\sum_{k=1}^{N/2} b_k \cos[\omega(k-\frac{1}{2})T]$
Antisymmetrical	Odd	$e^{-j[\omega(N-1)T/2-\pi/2]}\sum_{k=1}^{(N-1)/2} a_k \sin \omega kT$
	Even	$e^{-j[\omega(N-1)T/2-\pi/2]}\sum_{k=1}^{N/2} b_k \sin[\omega(k-\frac{1}{2})T]$

where $a_0 = h\left[\frac{(N-1)T}{2}\right]$, $a_k = 2h\left[\left(\frac{N-1}{2}-k\right)T\right]$, $b_k = 2h\left[\left(\frac{N}{2}-k\right)T\right]$

Location of Zeros

- The impulse response symmetry conditions described impose certain restrictions on the zeros of transfer function $H(z)$.

Location of Zeros

- The impulse response symmetry conditions described impose certain restrictions on the zeros of transfer function $H(z)$.
- For *odd* N, we can write

$$H(z) = \frac{1}{z^{(N-1)/2}} \left\{ \sum_{n=0}^{(N-3)/2} h(nT)\left(z^{(N-1)/2-n} \pm z^{-[(N-1)/2-n]}\right) \right.$$
$$\left. +\frac{1}{2}h\left[\frac{(N-1)T}{2}\right]\left(z^0 \pm z^0\right)\right\} \qquad \text{(C)}$$

where the negative sign applies to the case of antisymmetrical impulse response.

. . .

$$H(z) = \frac{1}{z^{(N-1)/2}} \left\{ \sum_{n=0}^{(N-3)/2} h(nT) \left(z^{(N-1)/2-n} \pm z^{-[(N-1)/2-n]} \right) + \frac{1}{2} h\left[\frac{(N-1)T}{2} \right] (z^0 \pm z^0) \right\} \qquad \text{(C)}$$

- With $(N-1)/2 - n = k$, Eq. (C) can be expressed as

$$H(z) = \frac{N(z)}{D(z)} = \frac{1}{z^{(N-1)/2}} \sum_{k=0}^{(N-1)/2} \frac{a_k}{2} (z^k \pm z^{-k})$$

where a_0 and a_k are given in the table of frequency responses shown earlier.

...

$$H(z) = \frac{N(z)}{D(z)} = \frac{1}{z^{(N-1)/2}} \sum_{k=0}^{(N-1)/2} \frac{a_k}{2}(z^k \pm z^{-k})$$

- The zeros of $H(z)$ are the roots of

$$N(z) = \sum_{k=0}^{(N-1)/2} a_k(z^k \pm z^{-k})$$

Location of Zeros *Cont'd*

...

$$H(z) = \frac{N(z)}{D(z)} = \frac{1}{z^{(N-1)/2}} \sum_{k=0}^{(N-1)/2} \frac{a_k}{2}(z^k \pm z^{-k})$$

- The zeros of $H(z)$ are the roots of

$$N(z) = \sum_{k=0}^{(N-1)/2} a_k(z^k \pm z^{-k})$$

- If we replace z by z^{-1} in $N(z)$, we get

$$\begin{aligned} N(z^{-1}) &= \sum_{k=0}^{(N-1)/2} a_k(z^{-k} \pm z^k) \\ &= \pm \sum_{k=0}^{(N-1)/2} a_k(z^k \pm z^{-k}) = \pm N(z) \end{aligned}$$

. . .

$$N(z^{-1}) = \pm N(z)$$

- The same relation holds for *even* N, as can be easily shown.

. . .

$$N(z^{-1}) = \pm N(z)$$

- The same relation holds for *even* N, as can be easily shown.
- Therefore, if $z_i = r_i e^{j\psi_i}$ is a zero of $H(z)$, then its *reciprocal* $z_i^{-1} = e^{-j\psi_i}/r_i$ must also be a zero of $H(z)$.

The property $N(z^{-1}) = \pm N(z)$ imposes the following constraints on the zeros of the transfer function:

1. An arbitrary number of zeros can be located at $z_i = \pm 1$ since $z_i^{-1} = \pm 1$.

The property $N(z^{-1}) = \pm N(z)$ imposes the following constraints on the zeros of the transfer function:

1. An arbitrary number of zeros can be located at $z_i = \pm 1$ since $z_i^{-1} = \pm 1$.
2. An arbitrary number of complex-conjugate pairs of zeros can be located on the unit circle since

$$(z - z_i)(z - z_i^*) = (z - e^{j\psi_i})(z - e^{-j\psi_i}) = \left(z - \frac{1}{z_i^*}\right)\left(z - \frac{1}{z_i}\right)$$

Location of Zeros *Cont'd*

The property $N(z^{-1}) = \pm N(z)$ imposes the following constraints on the zeros of the transfer function:

1. An arbitrary number of zeros can be located at $z_i = \pm 1$ since $z_i^{-1} = \pm 1$.
2. An arbitrary number of complex-conjugate pairs of zeros can be located on the unit circle since

$$(z - z_i)(z - z_i^*) = (z - e^{j\psi_i})(z - e^{-j\psi_i}) = \left(z - \frac{1}{z_i^*}\right)\left(z - \frac{1}{z_i}\right)$$

3. Real zeros off the unit circle must occur in reciprocal pairs.

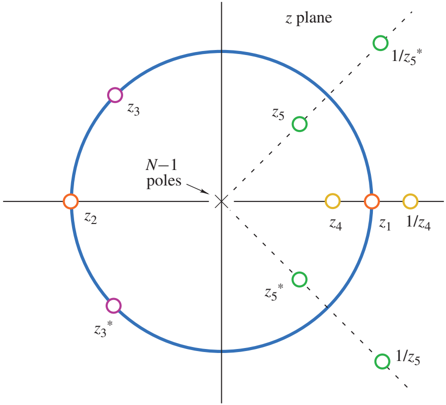
The property $N(z^{-1}) = \pm N(z)$ imposes the following constraints on the zeros of the transfer function:

1. An arbitrary number of zeros can be located at $z_i = \pm 1$ since $z_i^{-1} = \pm 1$.
2. An arbitrary number of complex-conjugate pairs of zeros can be located on the unit circle since

$$(z - z_i)(z - z_i^*) = (z - e^{j\psi_i})(z - e^{-j\psi_i}) = \left(z - \frac{1}{z_i^*}\right)\left(z - \frac{1}{z_i}\right)$$

3. Real zeros off the unit circle must occur in reciprocal pairs.
4. Complex zeros off the unit circle must occur in groups of four, namely, z_i, z_i^*, and their reciprocals.

Location of Zeros *Cont'd*



Note: Polynomials with these properties are called *mirror-image polynomials*.

This slide concludes the presentation.
Thank you for your attention.