Chapter 10
APPROXIMATIONS FOR ANALOG FILTERS

10.1 Introduction, 10.2 Realizability
10.3 to 10.7 Butterworth, Chebyshev, Inverse-Chebyshev,
Elliptic, and Bessel-Thomson Approximations

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Victoria, BC, Canada
Email: aantoniou@ieee.org

July 14, 2018
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- Butterworth,
- Chebyshev,
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This presentation deals with the basics of these approximations.
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An analog filter such as the one shown below can be represented by the equation

\[ \frac{V_o(s)}{V_i(s)} = H(s) = \frac{N(s)}{D(s)} \]

where

- \( V_i(s) \) is the Laplace transform of the input voltage \( v_i(t) \),
- \( V_o(s) \) is the Laplace transform of the output voltage \( v_o(t) \),
- \( H(s) \) is the transfer function,
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The loss (or attenuation) is defined as

\[ L(\omega^2) = \left| \frac{V_i(j\omega)}{V_o(j\omega)} \right|^2 = \left| \frac{V_i(j\omega)}{V_o(j\omega)} \right|^2 = \frac{1}{|H(j\omega)|^2} = 10\log \frac{1}{H(j\omega)H(-j\omega)} \]

Hence the loss in dB is given by

\[ A(\omega) = 10\log L(\omega^2) = 10\log \frac{1}{|H(j\omega)|^2} = -20\log |H(j\omega)| \]

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As a function of \( \omega \), \( A(\omega) \) is said to be the loss characteristic.
The phase shift and group delay of analog filters are defined just as in digital filters, namely, the phase shift is the phase angle of the frequency response and the group delay is the negative of the derivative of the phase angle with respect to $\omega$, i.e.,

$$\theta(\omega) = \arg H(j\omega) \quad \text{and} \quad \tau(\omega) = -\frac{d\theta(\omega)}{d\omega}$$
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As functions of $\omega$, $\theta(\omega)$ and $\tau(\omega)$ are the *phase response* and *delay characteristic*, respectively.
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If we replace \( \omega \) by \( s/j \) in \( L(\omega^2) \), we get the so-called \textit{loss function}

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\[ L(-s^2) = \frac{D(s)D(-s)}{N(s)N(-s)} \]

Thus if the transfer function of an analog filter is known, its loss function can be readily deduced.
If

\[ H(s) = \frac{N(s)}{D(s)} = \frac{\prod_{i=1}^{M}(s - z_i)}{\prod_{i=1}^{N}(s - p_i)} \]

then

\[ L(-s^2) = \frac{D(s)D(-s)}{N(s)N(-s)} = \frac{\prod_{i=1}^{N}(s - p_i)\prod_{i=1}^{N}(-s - p_i)}{\prod_{i=1}^{M}(s - z_i)\prod_{i=1}^{M}(-s - z_i)} \]

\[ = (-1)^{N-M} \frac{\prod_{i=1}^{N}(s - p_i)\prod_{i=1}^{N}[s - (-p_i)]}{\prod_{i=1}^{M}(s - z_i)\prod_{i=1}^{M}[s - (-z_i)]} \]
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Therefore,

- the zeros of the loss function are the poles of the transfer function and their negatives, and
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Zero-pole plots for transfer function and loss function:
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- The frequency range $0$ to $\omega_c$ is the **passband**.
- The frequency range $\omega_c$ to $\infty$ is the **stopband**.
- The boundary between the passband and stopband, namely, $\omega_c$, is the **cutoff frequency**.

\[ A(\omega) \]

![Graph](image)
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In the classical solutions of the approximation problem, an ideal *normalized* lowpass loss characteristic is assumed with a cutoff frequency of order unity, i.e., $\omega_c \approx 1$.

A set of formulas are then derived that yield the *zeros and poles* or *coefficients* of the transfer function for a specified filter order.
Classical approximations such as the Butterworth, Chebyshev, inverse-Chebyshev, and elliptic approximations lead to a loss characteristic where

- The loss is equal to or less than $A_p$ dB over the frequency range $0$ to $\omega_p$;
- The loss is equal to or greater than $A_a$ dB over the frequency range $\omega_a$ to $\infty$.

Parameters $\omega_p$ and $\omega_a$ are the passband and stopband edges, $A_p$ is the maximum passband loss (or attenuation), and $A_a$ is the minimum stopband loss (or attenuation), respectively.
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The quality of an approximation depends on the values of $A_p$ and $A_a$ for a given filter order, i.e., a lower $A_p$ and a larger $A_a$ correspond to a better filter.
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Filter denormalization can be applied through the use of a class of analog-filter transformations.

These transformations will be discussed in the next presentation.
Realizability Constraints

- *Realizability constraints* are constraints that must be satisfied by a transfer function in order to be realizable in terms of an analog-filter network.

  - The coefficients must be real. This requirement is imposed by the fact that inductances, capacitances, and resistances are required to be real quantities.
  - The degree of the numerator polynomial must be equal to or less than the degree of the denominator polynomial. Otherwise, the transfer function would represent a noncausal system which would not be realizable as a real-time analog filter.
  - The poles must be in the left half $s$ plane. Otherwise, the transfer function would represent an unstable system.
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In the slides that follow, the basic features of the various classical analog-filter approximations will be presented such as:

- The underlying assumptions in the derivation.
- Typical loss characteristics.
- Available independent parameters.
- Formula for the loss as a function of the independent parameters.
- Minimum filter order to achieve prescribed specifications.
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The *Butterworth approximation* is derived on the assumption that the loss function $L(-s^2)$ is a polynomial. Since

$$\lim_{s \to \infty} L(-s^2) = \lim_{\omega \to \infty} L(\omega^2) = a_0 + a_2\omega^2 + \cdots + a_{2n}\omega^{2n} \to \infty$$

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This is achieved by letting

\[
L(0) = 1, \quad \frac{d^k L(x)}{dx^k} \bigg|_{x=0} = 0 \quad \text{for} \quad k \leq n - 1
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where \( x = \omega^2 \), i.e., \( n \) derivatives of the loss are set to zero at zero frequency.
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where $x = \omega^2$, i.e., $n$ derivatives of the loss are set to zero at zero frequency.

- Assuming that $L(1) = 2$, the loss function in dB can be expressed as

$$L(\omega^2) = 1 + \omega^{2n} \quad \text{and} \quad A(\omega) = 10 \log(1 + \omega^{2n})$$
Butterworth Approximation Cont’d

- Typical loss characteristics:

\[ A(\omega), \text{dB} \]
\[ \omega, \text{rad/s} \]

- For different orders:
  - \( n = 3 \)
  - \( n = 6 \)
  - \( n = 9 \)
The loss function for the normalized Butterworth approximation (3-dB frequency at 1 rad/s) is given by

\[ L(-s^2) = 1 + (-s^2)^n = \prod_{i=1}^{2n} (s - z_i) \]

where

\[ z_i = \begin{cases} 
\exp\left(j\frac{(2i-1)\pi}{2n}\right) & \text{for even } n \\
\exp\left(j\frac{(i-1)\pi}{n}\right) & \text{for odd } n 
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Since \(|z_k| = 1\), the zeros of \(L(-s^2)\) are located on the unit circle \(|s| = 1|\).
Zero-pole plots for loss function:
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Therefore, they are identical with the zeros of the loss function located in the left-half $s$ plane.
Typically in practice, the required filter order is unknown.

For Butterworth, Chebyshev, inverse-Chebyshev, and elliptic filters, it can be easily deduced if the required specifications are known.

Let us assume that we need a normalized Butterworth filter with a maximum passband loss $A_p$, minimum stopband loss $A_a$, passband edge $\omega_p$, and stopband edge $\omega_a$.

The minimum filter order that will satisfy the required specifications must be large enough to satisfy both of the following inequalities:

$$n \geq \left\lceil -\log (10^{0.1 A_p} - 1) \right\rceil (\omega_p^{-2} \log)$$

and

$$n \geq \log (10^{0.1 A_a} - 1) 2 \log \omega_a$$

(See textbook for derivations and examples.)
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Butterworth Approximation Cont’d

\[ n \geq \frac{-\log(10^{0.1A_p} - 1)}{(-2 \log \omega_p)} \quad \text{and} \quad n \geq \frac{\log(10^{0.1A_a} - 1)}{2 \log \omega_a} \]

- The right-hand sides in the above inequalities will normally yield a mixed number but since the filter order must be an integer, the value obtained must be rounded up to the next integer.
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\bullet \bullet \bullet \quad n \geq \frac{-\log(10^{0.1A_p} - 1)}{(-2 \log \omega_p)} \quad \text{and} \quad n \geq \frac{\log(10^{0.1A_a} - 1)}{2 \log \omega_a}
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As a result, the required specifications will usually be slightly oversatisfied.
\[ n \geq \frac{[- \log (10^{0.1A_p} - 1)]}{(-2 \log \omega_p)} \quad \text{and} \quad n \geq \frac{\log (10^{0.1A_a} - 1)}{2 \log \omega_a} \]

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As a result, the required specifications will usually be slightly oversatisfied.

- Once the required filter order is determined, the actual maximum passband loss and minimum stopband loss can be calculated as

\[ A_p = A(\omega_p) = 10 \log (1 + \omega_p^{2n}) \quad \text{and} \quad A_a = A(\omega_a) = 10 \log (1 + \omega_a^{2n}) \]

respectively.
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Chebyshev Approximation

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- A more balanced characteristic with respect to the passband can be achieved by employing the Chebyshev approximation.
As in the Butterworth approximation, the loss function in the Chebyshev approximation is assumed to be a polynomial in $s$, which would assure a lowpass characteristic.
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On the basis of this assumption, a differential equation is constructed whose solution gives the zeros of the loss function.

Then by neglecting the zeros of the loss function in the right-half $s$ plane, the poles of the transfer function can be obtained.
In the case of a fourth-order Chebyshev filter the passband loss is assumed to be zero at $\omega = \Omega_{01}, \Omega_{02}$ and equal to $A_p$ at $\omega = 0, \hat{\Omega}_1, 1$ as shown in the figure:
On using all the information that can be extracted from the figure shown, a differential equation of the form

\[
\left( \frac{dF(\omega)}{d\omega} \right)^2 = \frac{M_4[1 - F^2(\omega)]}{1 - \omega^2}
\]

can be constructed.
On using all the information that can be extracted from the figure shown, a differential equation of the form

\[
\left( \frac{dF(\omega)}{d\omega} \right)^2 = \frac{M_4[1 - F^2(\omega)]}{1 - \omega^2}
\]

can be constructed.

The solution of this differential equation gives the loss as

\[
L(\omega^2) = 1 + \varepsilon^2 F^2(\omega)
\]

where

\[
\varepsilon^2 = 10^{0.1 A_p} - 1
\]

and

\[
F(\omega) = T_4(\omega) = \cos(4 \cos^{-1} \omega)
\]
On using all the information that can be extracted from the figure shown, a differential equation of the form

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and

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F(\omega) = T_4(\omega) = \cos(4 \cos^{-1} \omega)
\]

The function \(\cos(4 \cos^{-1} \omega)\) is actually a polynomial known as the 4th-order Chebyshev polynomial.
Similarly, for an $n$th-order Chebyshev approximation, one can show that

$$A(\omega) = 10 \log L(\omega^2) = 10 \log [1 + \varepsilon^2 T_n^2(\omega)]$$

where $\varepsilon^2 = 10^{0.1A_p} - 1$

and $T_n(\omega) = \begin{cases} 
\cos(n \cos^{-1} \omega) & \text{for } |\omega| \leq 1 \\
\cosh(n \cosh^{-1} \omega) & \text{for } |\omega| > 1 
\end{cases}$

is the $n$th-order Chebyshev polynomial.
Typical loss characteristics for Chebyshev approximation:

- Loss, dB
- \( \omega \), rad/s
- Loss, dB
- \( n = 4 \)
- \( n = 7 \)
The zeros of the loss function for a normalized \( n \)th-order Chebyshev approximation (\( \omega_p = 1 \) rad/s) are given by \( s_i = \sigma_i + j\omega_i \) where

\[
\sigma_i = \pm \sinh \left( \frac{1}{n} \sinh^{-1} \frac{1}{\varepsilon} \right) \sin \left( \frac{(2i - 1)\pi}{2n} \right)
\]

\[
\omega_i = \cosh \left( \frac{1}{n} \sinh^{-1} \frac{1}{\varepsilon} \right) \cos \left( \frac{(2i - 1)\pi}{2n} \right)
\]

for \( i = 1, 2, \ldots, n \).
The zeros of the loss function for a normalized $n$th-order Chebyshev approximation ($\omega_p = 1$ rad/s) are given by $s_i = \sigma_i + j\omega_i$ where

$$\sigma_i = \pm \sinh\left(\frac{1}{n} \sinh^{-1} \frac{1}{\varepsilon}\right) \sin \frac{(2i - 1)\pi}{2n}$$

$$\omega_i = \cosh\left(\frac{1}{n} \sinh^{-1} \frac{1}{\varepsilon}\right) \cos \frac{(2i - 1)\pi}{2n}$$

for $i = 1, 2, \ldots, n$.

From these equations, we note that

$$\frac{\sigma_i^2}{\sinh^2 u} + \frac{\omega_i^2}{\cosh^2 u} = 1 \quad \text{where} \quad u = \frac{1}{n} \sinh^{-1} \frac{1}{\varepsilon}$$

i.e., the zeros of $L(-s^2)$ are located on an ellipse.
Chebyshev Approximation \textit{Cont’d}

- Typical zero-pole plots for Chebyshev approximation:
  (a) $n = 5$ $A_p = 1$ dB; (b) $n = 6$ $A_p = 1$ dB.
An $n$th-order normalized Chebyshev transfer function with a passband edge $\omega_p = 1$ rad/s and a maximum passband loss of $A_p$ dB can be determined as follows:

$$H_N(s) = \frac{H_0}{D_0(s) \prod_i (s - p_i)(s - p_i^*)}$$

$$= \frac{H_0}{D_0(s) \prod_i [s^2 - 2\text{Re}(p_i)s + |p_i|^2]}$$

where

$$r = \begin{cases} \frac{n-1}{2} & \text{for odd } n \\ \frac{n}{2} & \text{for even } n \end{cases}$$

and

$$D_0(s) = \begin{cases} s - p_0 & \text{for odd } n \\ 1 & \text{for even } n \end{cases}$$
The poles and multiplier constant, $H_0$, can be calculated by using the following formulas in sequence:

\[ \varepsilon = \sqrt{10^{0.1A_p} - 1} \]

\[ p_0 = \sigma_{(n+1)/2} \quad \text{with} \quad \sigma_{(n+1)/2} = -\sinh\left(\frac{1}{n} \sinh^{-1} \frac{1}{\varepsilon}\right) \]

\[ p_i = \sigma_i + j\omega_i \quad \text{for} \quad i = 1, 2, \ldots, r \]

where

\[ \sigma_i = -\sinh\left(\frac{1}{n} \sinh^{-1} \frac{1}{\varepsilon}\right) \sin \left(2i - 1\right) \frac{\pi}{2n} \]

\[ \omega_i = \cosh\left(\frac{1}{n} \sinh^{-1} \frac{1}{\varepsilon}\right) \cos \left(2i - 1\right) \frac{\pi}{2n} \]

\[ H_0 = \begin{cases} -p_0 \prod_{i=1}^{r} |p_i|^2 & \text{for odd} \ n \\ 10^{-0.05A_p} \prod_{i=1}^{r} |p_i|^2 & \text{for even} \ n \end{cases} \]
The minimum filter order required to achieve a maximum passband loss of $A_p$ and a minimum stopband loss of $A_a$ must be large enough to satisfy the inequality

$$n \geq \frac{\cosh^{-1} \sqrt{D}}{\cosh^{-1} \omega_a} \quad \text{where} \quad D = \frac{10^{0.1A_a} - 1}{10^{0.1A_p} - 1}$$
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As in the Butterworth approximation, the value at the right-hand side of the inequality must be rounded up to the next integer. As a result, the minimum stopband loss will usually be slightly oversatisfied.
Chebyshev Approximation Cont’d

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The actual minimum stopband loss can be calculated as

$$A_a = A(\omega_a) = 10 \log L(\omega_a^2) = 10 \log[1 + \varepsilon^2 T_n^2(\omega_a)]$$
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In the Chebyshev approximation, the actual maximum passband loss will be exactly as specified, i.e., $A_p$. 
The *inverse-Chebyshev* approximation is closely related to the Chebyshev approximation, as may be expected, and it is actually derived from the Chebyshev approximation.
The *inverse-Chebyshev* approximation is closely related to the Chebyshev approximation, as may be expected, and it is actually derived from the Chebyshev approximation.

The passband loss in the inverse-Chebyshev is very similar to that of the Butterworth approximation, i.e., it is an increasing monotonic function of $\omega$, while the stopband loss oscillates between infinity and a prescribed minimum loss $A_a$. 
Typical loss characteristics for inverse-Chebyshev approximation:
The loss for the inverse-Chebyshev approximation is given by

\[ A(\omega) = 10 \log \left[ 1 + \frac{1}{\delta^2 T_n^2(1/\omega)} \right] \]

where

\[ \delta^2 = \frac{1}{10^{0.1A_a} - 1} \]

and the stopband extends from \( \omega = 1 \) to \( \infty \).
The *normalized* transfer function for a specified order, \( n \), stopband edge of \( \omega_a = 1 \text{ rad/s} \), and minimum stopband loss, \( A_a \), is given by

\[
H_N(s) = \frac{H_0}{D_0(s)} \prod_{i=1}^{r} \frac{(s - 1/z_i)(s - 1/z_i^*)}{(s - 1/p_i)(s - 1/p_i^*)}
\]

\[
= \frac{H_0}{D_0(s)} \prod_{i=1}^{r} \frac{s^2 + \frac{1}{|z_i|^2}}{s^2 - 2\text{Re} \left( \frac{1}{p_i} \right) s + \frac{1}{|p_i|^2}}
\]

where

\[
r = \begin{cases} 
\frac{n-1}{2} & \text{for odd } n \\
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\end{cases}
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\]

and

\[
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\]

The parameters of the transfer function can be calculated by using the formulas in the next slide.
Inverse-Chebyshev Approximation \textit{Cont'd}

\[ \delta = \frac{1}{\sqrt{10^{0.1}A_a - 1}}, \quad z_i = j \cos \left( \frac{2i - 1}{2n} \pi \right) \quad \text{for} \quad 1, 2, \ldots, r \]

\[ p_0 = \sigma_{(n+1)/2} \quad \text{with} \quad \sigma_{(n+1)/2} = -\sinh \left( \frac{1}{n} \sinh^{-1} \frac{1}{\delta} \right) \]

\[ p_i = \sigma_i + j\omega_i \quad \text{for} \quad 1, 2, \ldots, r \]

with \[ \sigma_i = -\sinh \left( \frac{1}{n} \sinh^{-1} \frac{1}{\delta} \right) \sin \left( \frac{2i - 1}{2n} \pi \right) \]

\[ \omega_i = \cosh \left( \frac{1}{n} \sinh^{-1} \frac{1}{\delta} \right) \cos \left( \frac{2i - 1}{2n} \pi \right) \]

and \[ H_0 = \begin{cases} \frac{1}{-p_0} \prod_{i=1}^{r} \frac{|z_i|^2}{|p_i|^2} & \text{for odd } n \\ \prod_{i=1}^{r} \frac{|z_i|^2}{|p_i|^2} & \text{for even } n \end{cases} \]
The minimum filter order required to achieve a maximum passband loss of $A_p$ and a minimum stopband loss of $A_a$ must be large enough to satisfy the inequality

$$n \geq \frac{\cosh^{-1} \sqrt{D}}{\cosh^{-1}(1/\omega_p)}$$

where

$$D = \frac{10^{0.1A_a} - 1}{10^{0.1A_p} - 1}$$
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The value of the right-hand side of the above inequality is rarely an integer and, therefore, it must be rounded up to the next integer. This will cause the actual maximum passband loss to be slightly oversatisfied.

The actual maximum passband loss can be calculated as

$$A_p(\omega_p) = 10 \log \left[ 1 + \frac{1}{\delta^2 T n(1/\omega_p)} \right]$$

where $\delta^2 = 10^{0.1A_a}$. In this approximation, the actual minimum stopband loss will be exactly as specified, i.e., $A_a$. 
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Elliptic Approximation

- The Chebyshev approximation yields a much better *passband* and the inverse-Chebyshev approximation yields a much better *stopband* than the Butterworth approximation.
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- The elliptic approximation is more efficient than all the other analog-filter approximations in that the transition between passband and stopband is steeper for a given approximation order.
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- The elliptic approximation is more efficient than all the other analog-filter approximations in that the transition between passband and stopband is steeper for a given approximation order.

In fact, this is the *optimal* approximation for a given piecewise constant approximation.
Loss characteristic for a 5th-order elliptic approximation:
The passband loss is assumed to oscillate between zero and a prescribed maximum $A_p$ and the stopband loss is assumed to oscillate between infinity and a prescribed minimum $A_a$. On the basis of the assumed structure of the loss characteristic, a differential equation is derived, as in the case of the Chebyshev approximation. After considerable mathematical complexity, the differential equation obtained is solved through the use of elliptic functions, and the parameters of the transfer function are deduced. The approximation owes its name to the use of elliptic functions in the derivation.
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• On the basis of the assumed structure of the loss characteristic, a differential equation is derived, as in the case of the Chebyshev approximation.

• After considerable mathematical complexity, the differential equation obtained is solved through the use of *elliptic functions*, and the parameters of the transfer function are deduced.
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After considerable mathematical complexity, the differential equation obtained is solved through the use of elliptic functions, and the parameters of the transfer function are deduced.

The approximation owes its name to the use of elliptic functions in the derivation.
The passband and stopband edges and cutoff frequency of a *normalized* elliptic approximation are defined as follows:

\[
\omega_p = \sqrt{k}, \quad \omega_a = \frac{1}{\sqrt{k}}, \quad \omega_c = \sqrt{\omega_a \omega_p} = 1
\]

 Constants \( k \) and \( k_1 \) given by

\[
k = \frac{\omega_p}{\omega_a} \quad \text{and} \quad k_1 = \left( \frac{10^{0.1A_p} - 1}{10^{0.1A_a} - 1} \right)^{1/2}
\]

are known as the *selectivity* and *discrimination* constants.
A normalized elliptic lowpass filter with a selectivity factor $k$, passband edge $\omega_p = \sqrt{k}$, stopband edge $\omega_a = 1/\sqrt{k}$, a maximum passband loss of $A_p$ dB, and a minimum stopband loss equal to or in excess of $A_a$ dB has a transfer function of the form

$$H_N(s) = \frac{H_0}{D_0(s)} \prod_{i=1}^{r} \frac{s^2 + a_{0i}}{s^2 + b_{1i}s + b_{0i}}$$

where

$$r = \begin{cases} \frac{n-1}{2} & \text{for odd } n \\ n/2 & \text{for even } n \end{cases}$$

and

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A normalized elliptic lowpass filter with a selectivity factor \( k \), passband edge \( \omega_p = \sqrt{k} \), stopband edge \( \omega_a = 1/\sqrt{k} \), a maximum passband loss of \( A_p \) dB, and a minimum stopband loss equal to or in excess of \( A_a \) dB has a transfer function of the form

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\frac{n-1}{2} & \text{for odd } n \\
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\]

and

\[
D_0(s) = \begin{cases} 
s + \sigma_0 & \text{for odd } n \\
1 & \text{for even } n 
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\]

The parameters of the transfer function can be obtained by using the formulas in the next three slides in sequence in the order shown.
Elliptic Approximation  *Cont’d*

\[ k' = \sqrt{1 - k^2} \]
\[ q_0 = \frac{1}{2} \left( \frac{1 - \sqrt{k'}}{1 + \sqrt{k'}} \right) \]
\[ q = q_0 + 2q_0^5 + 15q_0^9 + 150q_0^{13} \]
\[ D = \frac{10^{0.1A_a} - 1}{10^{0.1A_p} - 1} \]
\[ n \geq \frac{\log 16D}{\log(1/q)} \text{ (round up to the next integer)} \]
\[ \Lambda = \frac{1}{2n} \ln \frac{10^{0.05A_p} + 1}{10^{0.05A_p} - 1} \]
\[ \sigma_0 = \left| 2q^{1/4} \sum_{m=0}^{\infty} (-1)^m q^{m(m+1)} \sinh [(2m + 1)\Lambda] \right| \]
\[ \left| \frac{1 + 2 \sum_{m=1}^{\infty} (-1)^m q^{m^2} \cosh 2m\Lambda}{1 + 2 \sum_{m=1}^{\infty} (-1)^m q^{m^2} \cosh 2m\Lambda} \right| \]
Elliptic Approximation \textit{Cont’d}

\[ W = \sqrt{(1 + k\sigma_0^2) \left(1 + \frac{\sigma_0^2}{k}\right)} \]

\[ \Omega_i = \frac{2q^{1/4} \sum_{m=0}^{\infty} (-1)^m q^m(m+1) \sin \frac{(2m+1)\pi\mu}{n}}{1 + 2 \sum_{m=1}^{\infty} (-1)^m q^m \cos \frac{2m\pi\mu}{n}} \]

where \[ \mu = \begin{cases} i & \text{for odd } n \\ i - \frac{1}{2} & \text{for even } n \end{cases} \quad i = 1, 2, \ldots, r \]

\[ V_i = \sqrt{(1 - k\Omega_i^2) \left(1 - \frac{\Omega_i^2}{k}\right)} \]
Elliptic Approximation  Cont’d

\[ a_{0i} = \frac{1}{\Omega_i^2} \]

\[ b_{0i} = \frac{(\sigma_0 V_i)^2 + (\Omega_i W)^2}{(1 + \sigma_0^2 \Omega_i^2)^2} \]

\[ b_{1i} = \frac{2\sigma_0 V_i}{1 + \sigma_0^2 \Omega_i^2} \]

\[ H_0 = \begin{cases} \sigma_0 \prod_{i=1}^{r} \frac{b_{0i}}{a_{0i}} & \text{for odd } n \\ 10^{-0.05 A_p} \prod_{i=1}^{r} \frac{b_{0i}}{a_{0i}} & \text{for even } n \end{cases} \]
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$$A_a = A(\omega_a) = 10 \log \left( \frac{10^{0.1A_p} - 1}{16q^n} + 1 \right)$$
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The loss of an elliptic filter is usually calculated by using the transfer function, i.e.,

$$A(\omega) = 20 \log \left( \frac{1}{|H(j\omega)|} \right)$$
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(See textbook for details.)
Ideally, the group delay of a filter should be constant; equivalently, the phase shift should be a linear function of frequency to minimize delay distortion (see Sec. 5.7).

Since the only objective in the approximations described so far is to achieve a specific loss characteristic, there is no reason for the phase characteristic to turn out to be linear. In fact, it turns out to be highly nonlinear, as one might expect, particularly in the elliptic approximation.

The last approximation in Chap. 10, namely, the Bessel-Thomson approximation, is derived on the assumption that the group delay is maximally flat at zero frequency. As in the Butterworth and Chebyshev approximations, the loss function is a polynomial. Hence the Bessel-Thomson approximation is essentially a lowpass approximation.
Bessel-Thomson Approximation

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Since the only objective in the approximations described so far is to achieve a specific loss characteristic, there is no reason for the phase characteristic to turn out to be linear.

In fact, it turns out to be highly nonlinear, as one might expect, particularly in the elliptic approximation.
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Bessel-Thomson Approximation

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The transfer function for a \textit{normalized} Bessel-Thomson approximation is given by

$$H(s) = \frac{b_0}{\sum_{i=0}^{n} b_i s^i} = \frac{b_0}{s^n B(1/s)}$$

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- Function \( B(\cdot) \) is a *Bessel polynomial*, and \( s^n B(1/s) \) can be shown to have zeros in the left-half \( s \) plane, i.e., the *Bessel-Thomson approximation represents stable analog filters.*
Typical loss characteristics:
Typical delay characteristics:
This slide concludes the presentation. Thank you for your attention.