Chapter 11
DESIGN OF RECURSIVE (IIR) FILTERS
11.1 Introduction, 11.2 Realizability Constraints, 11.3 Invariant Impulse-Response Method, 11.4 Modified Invariant Impulse-Response Method, 11.5 Matched-z Transformation Method

Victoria, BC, Canada
Email: aantoniou@ieee.org

July 14, 2018

Introduction

- Approximation methods for the design of recursive (IIR) filters differ quite significantly from those used for the design of nonrecursive filters.

Introduction

- Approximation methods for the design of recursive (IIR) filters differ quite significantly from those used for the design of nonrecursive filters.
- The basic reason is that in recursive filters the transfer function is a ratio of polynomials of z whereas in nonrecursive filters it is a polynomial of negative powers of z.

Introduction

- Approximation methods for the design of recursive (IIR) filters differ quite significantly from those used for the design of nonrecursive filters.
- The basic reason is that in recursive filters the transfer function is a ratio of polynomials of z whereas in nonrecursive filters it is a polynomial of negative powers of z.
- In recursive filters, the approximation problem is usually solved through *indirect* or *direct* methods.

Introduction *Cont'd*

- In indirect methods, a continuous-time transfer function that satisfies certain specifications is first obtained by using one of the standard analog-filter approximations.

Introduction *Cont'd*

- In indirect methods, a continuous-time transfer function that satisfies certain specifications is first obtained by using one of the standard analog-filter approximations.
- The continuous-time transfer function obtained is then converted into a discrete-time transfer function.

Introduction *Cont'd*

- In indirect methods, a continuous-time transfer function that satisfies certain specifications is first obtained by using one of the standard analog-filter approximations.
- The continuous-time transfer function obtained is then converted into a discrete-time transfer function.
- In direct methods, the design problem is formulated as an optimization problem which is then solved using standard optimization methods.

Introduction *Cont'd*

- In indirect methods, a continuous-time transfer function that satisfies certain specifications is first obtained by using one of the standard analog-filter approximations.
- The continuous-time transfer function obtained is then converted into a discrete-time transfer function.
- In direct methods, the design problem is formulated as an optimization problem which is then solved using standard optimization methods.
- This presentation will deal with some indirect methods for the design of recursive filters.

Introduction *Cont'd*

- Several indirect approximation methods are available as follows:
  - Invariant impulse-response method
  - Modified invariant impulse-response method
  - Matched-z transformation method
  - Bilinear transformation method

Introduction *Cont'd*

- Several indirect approximation methods are available as follows:
  - Invariant impulse-response method
  - Modified invariant impulse-response method
  - Matched-z transformation method
  - Bilinear transformation method
- The most important among them are the *invariant impulse-response* and *bilinear transformation* methods.

Introduction *Cont'd*

- Several indirect approximation methods are available as follows:
  - Invariant impulse-response method
  - Modified invariant impulse-response method
  - Matched-z transformation method
  - Bilinear transformation method
- The most important among them are the *invariant impulse-response* and *bilinear transformation* methods.
- The first three are closely interrelated and will form the subject matter of this presentation.

Introduction *Cont'd*

- Several indirect approximation methods are available as follows:
  - Invariant impulse-response method
  - Modified invariant impulse-response method
  - Matched-z transformation method
  - Bilinear transformation method
- The most important among them are the *invariant impulse-response* and *bilinear transformation* methods.
- The first three are closely interrelated and will form the subject matter of this presentation.

  The bilinear transformation method will be discussed in the next presentation.

Realizability Constraints

- Before we discuss the available approximation methods for recursive filters, it is important to mention the constraints that are imposed on the transfer function, which are as follows:

Realizability Constraints

- Before we discuss the available approximation methods for recursive filters, it is important to mention the constraints that are imposed on the transfer function, which are as follows:
  - It must be a rational function of z with real coefficients. This follows from the fact that digital filters comprise unit delays, adders, and multipliers with real multiplier constants.

Realizability Constraints

- Before we discuss the available approximation methods for recursive filters, it is important to mention the constraints that are imposed on the transfer function, which are as follows:
  - It must be a rational function of z with real coefficients. This follows from the fact that digital filters comprise unit delays, adders, and multipliers with real multiplier constants.
  - Its poles must lie within the unit circle of the z plane to ensure that the filter is stable.

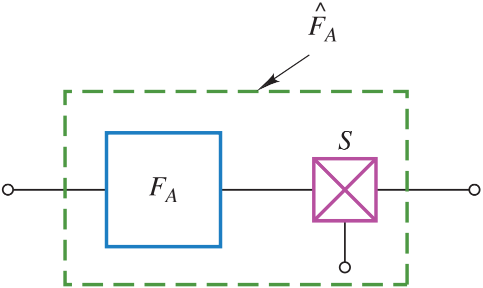
Realizability Constraints

- Before we discuss the available approximation methods for recursive filters, it is important to mention the constraints that are imposed on the transfer function, which are as follows:
  - It must be a rational function of z with real coefficients. This follows from the fact that digital filters comprise unit delays, adders, and multipliers with real multiplier constants.
  - Its poles must lie within the unit circle of the z plane to ensure that the filter is stable.
  - The degree of the numerator polynomial must be equal to or less than the degree of the denominator polynomial to ensure that the filter is causal.

Realizability Constraints

- Before we discuss the available approximation methods for recursive filters, it is important to mention the constraints that are imposed on the transfer function, which are as follows:
  - It must be a rational function of z with real coefficients. This follows from the fact that digital filters comprise unit delays, adders, and multipliers with real multiplier constants.
  - Its poles must lie within the unit circle of the z plane to ensure that the filter is stable.
  - The degree of the numerator polynomial must be equal to or less than the degree of the denominator polynomial to ensure that the filter is causal.
- These constraints will ensure that the transfer function is realizable in the form of a stable digital-filter network and are, therefore, said to be the *realizability* constraints.

- Given an analog filter, a corresponding digital filter can be obtained by constructing an impulse-modulated filter $\hat{F}_A$ as shown in the figure where S is an ideal impulse modulator and F_A is an analog filter characterized by a continuous-time transfer function $H_A(s)$.



- On the basis of the Poisson summation formula (see Chap. 6), the impulse-modulated filter can be represented by a continuous-time transfer function $\hat{H}_A(s)$ or, equivalently, by a discrete-time transfer function $H_D(z)$ as follows:

$$\hat{H}_A(j\omega) = H_D(e^{j\omega T}) = \frac{h_A(0+)}{2} + \frac{1}{T}\sum_{k=-\infty}^{\infty} H_A(j\omega + jk\omega_s)$$

where

$$h_A(t) = \mathcal{L}^{-1}H_A(s), \quad h_A(0+) = \lim_{s\to\infty}[sH_A(s)],$$

and

$$H_D(z) = \mathcal{Z}h_A(nT)$$

- Therefore, given an analog filter, a corresponding digital filter can be obtained as follows:

- Therefore, given an analog filter, a corresponding digital filter can be obtained as follows:
  - Deduce the impulse response of the analog filter as

$$h_A(t) = \mathcal{L}^{-1} H_A(s)$$

- Therefore, given an analog filter, a corresponding digital filter can be obtained as follows:
  - Deduce the impulse response of the analog filter as

$$h_A(t) = \mathcal{L}^{-1} H_A(s)$$

  - Replace t by nT in $h_A(t)$ to obtain $h_A(nT)$.

Invariant Impulse-Response Method *Cont'd*

- Therefore, given an analog filter, a corresponding digital filter can be obtained as follows:
  - Deduce the impulse response of the analog filter as

$$h_A(t) = \mathcal{L}^{-1} H_A(s)$$

  - Replace t by nT in $h_A(t)$ to obtain $h_A(nT)$.
  - Obtain the z transform of $h_A(nT)$, i.e.,

$$H_D(z) = \mathcal{Z} h_A(nT)$$

Invariant Impulse-Response Method *Cont'd*

- Therefore, given an analog filter, a corresponding digital filter can be obtained as follows:
  - Deduce the impulse response of the analog filter as

$$h_A(t) = \mathcal{L}^{-1} H_A(s)$$

  - Replace t by nT in $h_A(t)$ to obtain $h_A(nT)$.
  - Obtain the z transform of $h_A(nT)$, i.e.,

$$H_D(z) = \mathcal{Z} h_A(nT)$$

- The method is referred to as the *invariant-impulse response* method because the impulse response of the digital filter is exactly equal to the impulse response of the analog filter at $t = nT$ for $n = 0, 1, \ldots, \infty$.

- Obviously, we have a way of generating a discrete-time transfer function from a given continuous-time transfer function and, therefore, we can design a digital filter starting with an analog filter.

Invariant Impulse-Response Method *Cont'd*

- Obviously, we have a way of generating a discrete-time transfer function from a given continuous-time transfer function and, therefore, we can design a digital filter starting with an analog filter.
- However, the following important questions will immediately arise:

Invariant Impulse-Response Method *Cont'd*

- Obviously, we have a way of generating a discrete-time transfer function from a given continuous-time transfer function and, therefore, we can design a digital filter starting with an analog filter.
- However, the following important questions will immediately arise:
  1. How is the frequency response of the derived digital filter related to that of the analog filter?

Invariant Impulse-Response Method *Cont'd*

- Obviously, we have a way of generating a discrete-time transfer function from a given continuous-time transfer function and, therefore, we can design a digital filter starting with an analog filter.
- However, the following important questions will immediately arise:
  1. How is the frequency response of the derived digital filter related to that of the analog filter?
  2. Would the digital filter obtained be stable if the analog filter is stable?

- Obviously, we have a way of generating a discrete-time transfer function from a given continuous-time transfer function and, therefore, we can design a digital filter starting with an analog filter.
- However, the following important questions will immediately arise:
  1. How is the frequency response of the derived digital filter related to that of the analog filter?
  2. Would the digital filter obtained be stable if the analog filter is stable?
  3. Would the digital filter obtained be causal?

- Obviously, we have a way of generating a discrete-time transfer function from a given continuous-time transfer function and, therefore, we can design a digital filter starting with an analog filter.
- However, the following important questions will immediately arise:
  1. How is the frequency response of the derived digital filter related to that of the analog filter?
  2. Would the digital filter obtained be stable if the analog filter is stable?
  3. Would the digital filter obtained be causal?
  4. Would the discrete-time transfer function obtained have real coefficients?

- Let us examine the issue of the frequency response.

- Let us examine the issue of the frequency response.

  As was mentioned earlier, the Poisson summation formula gives the frequency response of the derived digital filter as

$$\hat{H}_A(j\omega) = H_D(e^{j\omega T}) = \frac{h(0+)}{2} + \frac{1}{T} \sum_{k=-\infty}^{\infty} H_A(j\omega + jk\omega_s)$$

- Let us examine the issue of the frequency response.

  As was mentioned earlier, the Poisson summation formula gives the frequency response of the derived digital filter as

$$\hat{H}_A(j\omega) = H_D(e^{j\omega T}) = \frac{h(0+)}{2} + \frac{1}{T}\sum_{k=-\infty}^{\infty} H_A(j\omega + jk\omega_s)$$

- If

$$H_A(j\omega) \approx 0 \quad \text{for} \quad |\omega| \geq \frac{\omega_s}{2}$$

  then

$$\sum_{k=-\infty, k\neq 0}^{\infty} H_A(j\omega + jk\omega_s) \approx 0 \quad \text{for} \quad |\omega| < \frac{\omega_s}{2}$$

  i.e., the side-bands contribute a negligible amount of *aliasing* error.

Invariant Impulse-Response Method *Cont'd*

- If, in addition, $h_A(0+) = 0$ then

$$\hat{H}_A(j\omega) = H_D(e^{j\omega T}) \approx \frac{1}{T} H_A(j\omega) \quad \text{for} \quad |\omega| < \frac{\omega_s}{2}$$

Invariant Impulse-Response Method *Cont'd*

- If, in addition, $h_A(0+) = 0$ then

$$\hat{H}_A(j\omega) = H_D(e^{j\omega T}) \approx \frac{1}{T} H_A(j\omega) \quad \text{for} \quad |\omega| < \frac{\omega_s}{2}$$

- In effect, given an analog filter that satisfies the stated bandlimiting conditions, a digital filter can be derived which would have *approximately the same frequency response* as the analog filter to within a multiplier constant $1/T$.

Invariant Impulse-Response Method *Cont'd*

- If, in addition, $h_A(0+) = 0$ then

$$\hat{H}_A(j\omega) = H_D(e^{j\omega T}) \approx \frac{1}{T} H_A(j\omega) \quad \text{for} \quad |\omega| < \frac{\omega_s}{2}$$

- In effect, given an analog filter that satisfies the stated bandlimiting conditions, a digital filter can be derived which would have *approximately the same frequency response* as the analog filter to within a multiplier constant $1/T$.

- Since

$$h_A(0+) = \lim_{s \to \infty} [sH_A(s)],$$

the *condition $h_A(0+) = 0$ is satisfied* if the denominator degree in $H_A(s)$ exceeds the numerator degree by at least two.

Invariant Impulse-Response Method *Cont'd*

- If, in addition, $h_A(0+) = 0$ then

$$\hat{H}_A(j\omega) = H_D(e^{j\omega T}) \approx \frac{1}{T} H_A(j\omega) \quad \text{for} \quad |\omega| < \frac{\omega_s}{2}$$

- In effect, given an analog filter that satisfies the stated bandlimiting conditions, a digital filter can be derived which would have *approximately the same frequency response* as the analog filter to within a multiplier constant $1/T$.

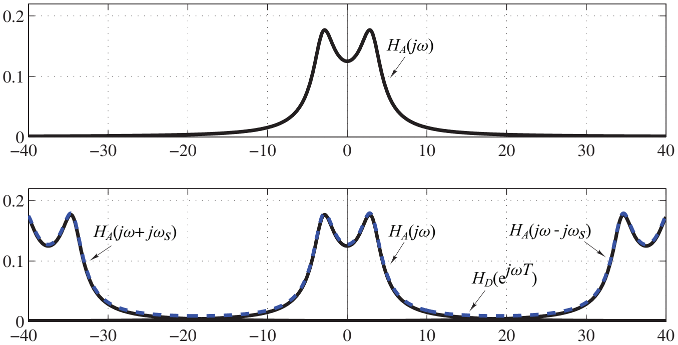
- Since

$$h_A(0+) = \lim_{s \to \infty} [sH_A(s)],$$

the *condition $h_A(0+) = 0$ is satisfied* if the denominator degree in $H_A(s)$ exceeds the numerator degree by at least two.

- The *multiplier constant $1/T$ can be eliminated* by multiplying the discrete-time transfer function obtained, $H_D(z)$, by T.

Invariant Impulse-Response Method *Cont'd*



Design Procedure

Given an analog filter characterized by a transfer function $H_A(s)$ *that satisfies the stated bandlimiting conditions*, a digital filter can be obtained by applying the following design procedure:

1. If the transfer function $H_A(s)$ is given in terms of its coefficients, i.e.,

$$H_A(s) = \frac{\sum_{i=0}^{M} a_i s^i}{\sum_{i=0}^{N} b_i s^i}$$

express it in terms of its *zeros and poles* as

$$H_A(s) = H_0 \frac{\prod_{i=1}^{M}(s - z_i)}{\prod_{i=1}^{N}(s - p_i)}$$

...

$$H_A(s) = H_0 \frac{\prod_{i=1}^{M}(s - z_i)}{\prod_{i=1}^{N}(s - p_i)}$$

2. Express the transfer function in terms of *partial fractions* as

$$H_A(s) = \sum_{i=1}^{N} \frac{A_i}{s - p_i}$$

...

$$H_A(s) = H_0 \frac{\prod_{i=1}^{M}(s - z_i)}{\prod_{i=1}^{N}(s - p_i)}$$

2. Express the transfer function in terms of *partial fractions* as

$$H_A(s) = \sum_{i=1}^{N} \frac{A_i}{s - p_i}$$

3. Deduce the *impulse response* of the analog filter as follows:

$$h_A(t) = \mathcal{L}^{-1} H_A(s) = \sum_{i=1}^{N} A_i e^{p_i t}$$

Design Procedure *Cont'd*

...

$$h_A(t) = \sum_{i=1}^{N} A_i e^{p_i t}$$

4. Replace t by nT in $h_A(t)$ to obtain $h_A(nT)$ as

$$h_A(nT) = \sum_{i=1}^{N} A_i e^{p_i nT}$$

. . .

$$h_A(t) = \sum_{i=1}^{N} A_i e^{p_i t}$$

4. Replace t by nT in $h_A(t)$ to obtain $h_A(nT)$ as

$$h_A(nT) = \sum_{i=1}^{N} A_i e^{p_i nT}$$

5. Obtain the z transform of $h_A(nT)$ as

$$H_D(z) = \mathcal{Z} h_A(nT) = \sum_{i=1}^{N} \frac{A_i z}{z - e^{Tp_i}}$$

...

$$H_D(z) = \sum_{i=1}^{N} \frac{A_i z}{z - e^{Tp_i}}$$

6. Multiply the discrete-time transfer function obtained in Step 5 by T, i.e.,

$$H'_D(z) = TH_D(z)$$

...

$$H_D(z) = \sum_{i=1}^{N} \frac{A_i z}{z - e^{Tp_i}}$$

6. Multiply the discrete-time transfer function obtained in Step 5 by T, i.e.,

$$H'_D(z) = TH_D(z)$$

7. Combine partial fractions with complex conjugate poles to obtain the modified discrete-time transfer function

$$H'_D(z) = \sum_{i=1}^{N/2} \frac{a_{1i} z + a_{2i} z^2}{b_{0i} + b_{1i} z + b_{2i} z^2} \quad \text{for even } N$$

or

$$H'_D(z) = \frac{a_{11} z}{b_{01} + z} + \sum_{i=2}^{(N-1)/2} \frac{a_{1i} z + a_{2i} z^2}{b_{0i} + b_{1i} z + b_{2i} z^2} \quad \text{for odd } N$$

Notes

- The bandlimiting condition implies that the numerator degree in $H_A(s)$ is less than the denominator degree.

  Consequently, there is no constant term in the partial fraction expansion of Step 2.

Notes

- The bandlimiting condition implies that the numerator degree in $H_A(s)$ is less than the denominator degree.

  Consequently, there is no constant term in the partial fraction expansion of Step 2.

- In Step 5, we note that a pole $p_i = \sigma_i + j\omega_i$ in the analog filter will yield a pole $\bar{p}_i$ in the digital filter where

$$\bar{p}_i = e^{Tp_i} = e^{T(\sigma_i + j\omega_i)} = e^{\sigma_i T} \cdot e^{j\omega_i T}$$

Notes

- The bandlimiting condition implies that the numerator degree in $H_A(s)$ is less than the denominator degree.

  Consequently, there is no constant term in the partial fraction expansion of Step 2.

- In Step 5, we note that a pole $p_i = \sigma_i + j\omega_i$ in the analog filter will yield a pole $\bar{p}_i$ in the digital filter where

  $$\bar{p}_i = e^{Tp_i} = e^{T(\sigma_i + j\omega_i)} = e^{\sigma_i T} \cdot e^{j\omega_i T}$$

  Hence $|\bar{p}_i| = e^{\sigma_i T}$ and if $\sigma_i < 0$ then $|\bar{p}_i| < 1$.

Notes

- The bandlimiting condition implies that the numerator degree in $H_A(s)$ is less than the denominator degree.

  Consequently, there is no constant term in the partial fraction expansion of Step 2.

- In Step 5, we note that a pole $p_i = \sigma_i + j\omega_i$ in the analog filter will yield a pole $\bar{p}_i$ in the digital filter where

$$\bar{p}_i = e^{Tp_i} = e^{T(\sigma_i + j\omega_i)} = e^{\sigma_i T} \cdot e^{j\omega_i T}$$

  Hence $|\bar{p}_i| = e^{\sigma_i T}$ and if $\sigma_i < 0$ then $|\bar{p}_i| < 1$.

  Therefore, if the analog filter has poles in the left-half s plane, the poles of the digital filter obtained will be located in the unit circle of the z plane.

Notes

- The bandlimiting condition implies that the numerator degree in $H_A(s)$ is less than the denominator degree.

  Consequently, there is no constant term in the partial fraction expansion of Step 2.

- In Step 5, we note that a pole $p_i = \sigma_i + j\omega_i$ in the analog filter will yield a pole $\bar{p}_i$ in the digital filter where

$$\bar{p}_i = e^{Tp_i} = e^{T(\sigma_i + j\omega_i)} = e^{\sigma_i T} \cdot e^{j\omega_i T}$$

  Hence $|\bar{p}_i| = e^{\sigma_i T}$ and if $\sigma_i < 0$ then $|\bar{p}_i| < 1$.

  Therefore, if the analog filter has poles in the left-half s plane, the poles of the digital filter obtained will be located in the unit circle of the z plane.

  That is, *a stable analog filter will yield a stable digital filter*.

Notes *Cont'd*

- In Step 6, we multiply the discrete-time transfer function obtained in Step 5 by T in order to cancel the multiplier constant $1/T$ introduced by the Poisson summation formula.

Notes *Cont'd*

- In Step 6, we multiply the discrete-time transfer function obtained in Step 5 by T in order to cancel the multiplier constant $1/T$ introduced by the Poisson summation formula.

  Thus we obtain a digital filter that has a baseband frequency response which is *approximately the same* as the frequency response of the analog filter.

- In Step 6, we multiply the discrete-time transfer function obtained in Step 5 by T in order to cancel the multiplier constant $1/T$ introduced by the Poisson summation formula.

  Thus we obtain a digital filter that has a baseband frequency response which is *approximately the same* as the frequency response of the analog filter.

- In Step 7, the coefficients of

$$H'_D(z) = \sum_{i=1}^{K} \frac{a_{1i}z + a_{2i}z^2}{b_{0i} + b_{1i}z + b_{2i}z^2}$$

  turn out to be *real* as required.

Notes *Cont'd*

- In Step 6, we multiply the discrete-time transfer function obtained in Step 5 by T in order to cancel the multiplier constant $1/T$ introduced by the Poisson summation formula.

  Thus we obtain a digital filter that has a baseband frequency response which is *approximately the same* as the frequency response of the analog filter.

- In Step 7, the coefficients of

$$H'_D(z) = \sum_{i=1}^{K} \frac{a_{1i}z + a_{2i}z^2}{b_{0i} + b_{1i}z + b_{2i}z^2}$$

  turn out to be *real* as required.

  This is due to the fact that *complex conjugate poles give complex conjugate residues*.

- If the transfer function is of odd order, the continuous-time transfer function $H_A(s)$ has a real pole which will produce a first-order partial transfer function in Step 7.

Notes *Cont'd*

- If the transfer function is of odd order, the continuous-time transfer function $H_A(s)$ has a real pole which will produce a first-order partial transfer function in Step 7.

- Each of the partial transfer functions in Step 7 represents a causal filter since the numerator degree does not exceed the denominator degree in each case.

  Consequently, $H'_D(z)$ also represents a *causal filter*.

- If the transfer function is of odd order, the continuous-time transfer function $H_A(s)$ has a real pole which will produce a first-order partial transfer function in Step 7.
- Each of the partial transfer functions in Step 7 represents a causal filter since the numerator degree does not exceed the denominator degree in each case.

  Consequently, $H'_D(z)$ also represents a *causal filter*.
- A filter designed by using the invariant-impulse-response method can be conveniently realized by using the *parallel realization* since the overall transfer function is a sum of first- or second-order transfer functions.

Features

- The method gives good results for *lowpass and bandpass Butterworth, Bessel, and Chebyshev filters*.

Features

- The method gives good results for *lowpass and bandpass Butterworth, Bessel, and Chebyshev filters*.

  This is because these filters have a *bandlimited frequency response*.

Features

- The method gives good results for *lowpass and bandpass Butterworth, Bessel, and Chebyshev filters*.

  This is because these filters have a *bandlimited frequency response*.

- The results are not so good for lowpass and bandpass elliptic filters.

Features

- The method gives good results for *lowpass and bandpass Butterworth, Bessel, and Chebyshev filters*.

  This is because these filters have a *bandlimited frequency response*.

- The results are not so good for lowpass and bandpass elliptic filters.

  In these filters, the frequency response is not bandlimited since the stopband gain oscillates between zero and a specified maximum.

Features

- The method gives good results for *lowpass and bandpass Butterworth, Bessel, and Chebyshev filters*.

  This is because these filters have a *bandlimited frequency response*.

- The results are not so good for lowpass and bandpass elliptic filters.

  In these filters, the frequency response is not bandlimited since the stopband gain oscillates between zero and a specified maximum.

- It *does not work* at all for highpass filters since these filters are not bandlimited by definition.

- If the bandlimiting condition is satisfied, then the frequency response is preserved which includes *both the amplitude response and phase responses*.

- If the bandlimiting condition is satisfied, then the frequency response is preserved which includes *both the amplitude response and phase responses*.

  Therefore, the *method tends to preserve the phase response* of the analog filter.

Features *Cont'd*

- If the bandlimiting condition is satisfied, then the frequency response is preserved which includes *both the amplitude response and phase responses*.

  Therefore, the *method tends to preserve the phase response* of the analog filter.

  Thus if a linear-phase continuous-time transfer function is used (e.g., Bessel-Thomson), a *linear-phase* digital filter is obtained.

Features *Cont'd*

- If the bandlimiting condition is satisfied, then the frequency response is preserved which includes *both the amplitude response and phase responses*.

  Therefore, the *method tends to preserve the phase response* of the analog filter.

  Thus if a linear-phase continuous-time transfer function is used (e.g., Bessel-Thomson), a *linear-phase* digital filter is obtained.

- To apply the method, all one needs to do is to find the residues of the continuous-time transfer function and calculate the poles of the discrete-time transfer function using the poles of the continuous-time transfer function.

- If the bandlimiting condition is satisfied, then the frequency response is preserved which includes *both the amplitude response and phase responses*.

  Therefore, the *method tends to preserve the phase response* of the analog filter.

  Thus if a linear-phase continuous-time transfer function is used (e.g., Bessel-Thomson), a *linear-phase* digital filter is obtained.

- To apply the method, all one needs to do is to find the residues of the continuous-time transfer function and calculate the poles of the discrete-time transfer function using the poles of the continuous-time transfer function.

  That is, the method is relatively *simple to apply*.

Example

Starting with a third-order normalized lowpass Chebyshev transfer function, obtain a discrete-time transfer function using the invariant-impulse response method.

Assume a maximum passband loss $A_p = 1.0$ dB and a sampling frequency $\omega_s = 10.0$ rad/s.

Example

Starting with a third-order normalized lowpass Chebyshev transfer function, obtain a discrete-time transfer function using the invariant-impulse response method.

Assume a maximum passband loss $A_p = 1.0$ dB and a sampling frequency $\omega_s = 10.0$ rad/s.

Solution The required Chebyshev transfer function can be readily obtained as

$$H_A(s) = \frac{H_0}{(s - p_1)(s - p_2)(s - p_2^*)}$$

where

$$H_0 = 0.4913, \; p_1 = -0.4942, \quad \text{and} \quad p_2, \; p_2^* = -0.2471 \pm j0.9660$$

(See Chap. 10.)

Example *Cont'd*

On expanding $H_A(s)$ into partial fractions as in Step 2 of the design procedure, we obtain

$$H_A(s) = \frac{A_1}{s - p_1} + \frac{A_2}{s - p_2} + \frac{A_3}{s - p_2^*}$$

where

$$\begin{aligned} A_1 &= \left. \frac{H_0}{(s - p_2)(s - p_2^*)} \right|_{s=p_1} = \frac{H_0}{(p_1 - p_2)(p_1 - p_2^*)} = 0.4942 \\ A_2 &= \left. \frac{H_0}{(s - p_1)(s - p_2^*)} \right|_{s=p_2} = \frac{H_0}{(p_2 - p_1)(p_2 - p_2^*)} \\ &= -0.2471 - j0.0632 \\ A_3 &= A_2^* = -0.2471 + j0.0632 \end{aligned}$$

...

$$H_A(s) = \frac{A_1}{s - p_1} + \frac{A_2}{s - p_2} + \frac{A_3}{s - p_2^*}$$

From Step 3, the impulse response of the analog filter is obtained as

$$h_A(t) = \mathcal{L}^{-1} H_A(s) = A_1 e^{p_1 t} + A_2 e^{p_2 t} + A_2^* e^{p_2 * t}$$

...

$$H_A(s) = \frac{A_1}{s - p_1} + \frac{A_2}{s - p_2} + \frac{A_3}{s - p_2^*}$$

From Step 3, the impulse response of the analog filter is obtained as

$$h_A(t) = \mathcal{L}^{-1} H_A(s) = A_1 e^{p_1 t} + A_2 e^{p_2 t} + A_2^* e^{p_2 * t}$$

Replacing t by nT in $h_A(t)$, as in Step 4, gives

$$h_A(nT) = A_1 e^{p_1 nT} + A_2 e^{p_2 nT} + A_2^* e^{p_2^* nT}$$

where $T = 2\pi/\omega_s = 2\pi/10.0 = 0.6283$ s.

...

$$h_A(nT) = A_1 e^{p_1 nT} + A_2 e^{p_2 nT} + A_2^* e^{p_2^* nT}$$

On applying the z transform to $h_A(nT)$, as in Step 5, we get the discrete-time transfer function as

$$H_D(z) = \mathcal{Z} h_A(nT) = \frac{A_1 z}{z - e^{Tp_1}} + \frac{A_2 z}{z - e^{Tp_2}} + \frac{A_2^* z}{z - e^{Tp_2^*}}$$

Example *Cont'd*

Now on multiplying $H_D(z)$ by T (to adjust the gain of the filter) and then combining partial fractions with complex conjugate poles, the discrete-time transfer function can be expressed as

$$\begin{aligned}
H'_D(z) &= \frac{A_1 z}{z - e^{Tp_1}} + \left(\frac{A_2 z}{z - e^{Tp_2}} + \frac{A_2^* z}{z - e^{Tp_3}} \right) \\
&= \frac{A_1 z}{z - e^{p_1 T}} + \frac{(A_2 + A_2^*)z^2 - (A_2 e^{p_2^* T} + A_2^* e^{p_2 T})z}{z^2 - (e^{p_2 T} + e^{p_2^* T})z + e^{p_2 T} \cdot e^{p_2^* T}} \\
&= \frac{A_1 z}{z - e^{p_1 T}} + \frac{2\text{Re}\,(A_2)z^2 - 2\text{Re}\,(A_2 e^{p_2^* T})z}{z^2 - 2\text{Re}\,(e^{p_2 T})z + |e^{p_2 T}|^2} \\
&= \frac{a_{11} z}{z + b_{01}} + \frac{a_{22} z^2 + a_{12} z}{z^2 + b_{12} z + b_{02}}
\end{aligned}$$

Example *Cont'd*

...

$$H'_D(z) = \frac{a_{11}z}{z + b_{01}} + \frac{a_{22}z^2 + a_{12}z}{z^2 + b_{12}z + b_{02}} \quad \blacksquare$$

where

$$\begin{aligned}
a_{11} &= A_1 = 0.3105 \\
b_{01} &= -e^{Tp_1} = -0.7331 \\
a_{22} &= 2\text{Re}\,(A_2) = -0.4942 \\
a_{12} &= -2\text{Re}\,(A_2 e^{p_2^* T}) = 0.4093 \\
b_{12} &= -2\text{Re}\,(e^{p_2 T}) = -1.4065 \\
b_{02} &= |e^{p_2 T}|^2 = 0.7331
\end{aligned}$$

Example

Design a digital filter by applying the invariant impulse-response method to the Bessel-Thomson transfer function

$$H_A(s) = \frac{105}{105 + 105s + 45s^2 + 10s^3 + s^4}$$

Employ a sampling frequency $\omega_s = 8$ rad/s; repeat with $\omega_s = 16$ rad/s.

Example

Design a digital filter by applying the invariant impulse-response method to the Bessel-Thomson transfer function

$$H_A(s) = \frac{105}{105 + 105s + 45s^2 + 10s^3 + s^4}$$

Employ a sampling frequency $\omega_s = 8$ rad/s; repeat with $\omega_s = 16$ rad/s.

Solution The poles and residues of $H_A(s)$ are given by

$$\begin{aligned} p_1, p_1^* &= -2.896211 \pm j0.8672341 \\ p_2, p_2^* &= -2.103789 \pm j2.657418 \\ R_1, R_1^* &= 1.663392 \mp j8.396299 \\ R_2, R_2^* &= -1.663392 \pm j2.244076 \end{aligned}$$

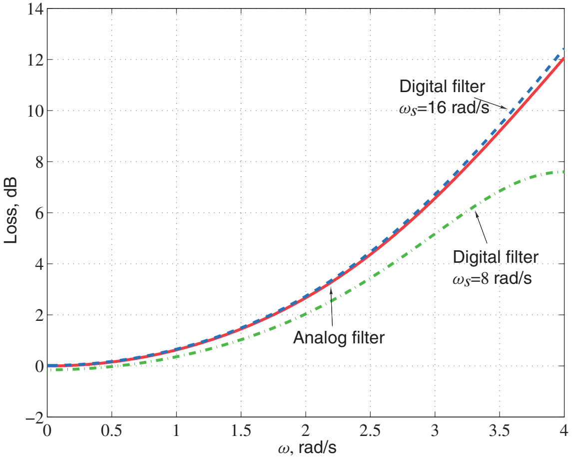
Example *Cont'd*

Steps 1 to 7 of the design procedure yield

$$TH_D(z) = \sum_{j=1}^{2} \frac{a_{1j}z + a_{2j}z^2}{b_{0j} + b_{1j}z + z^2}$$

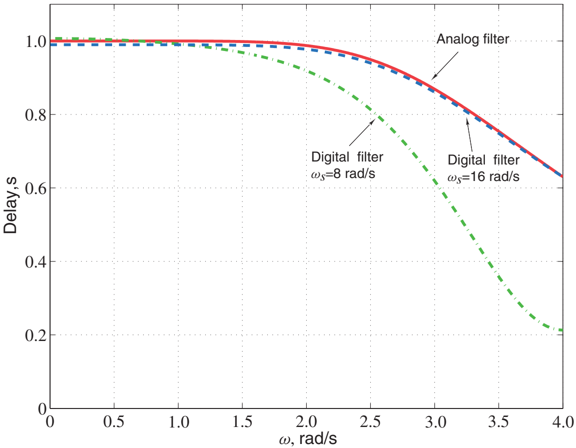
ω_s	j	a_{1j}	a_{2j}	b_{0j}	b_{1j}
8	1	6.452333E-1	2.612851	1.057399E-2	-1.597700E-1
	2	-8.345233E-1	-2.612851	3.671301E-2	1.891907E-1
16	1	3.114550E-1	1.306425	1.028299E-1	-6.045080E-1
	2	-3.790011E-1	-1.306425	1.916064E-1	-4.404794E-1

- Loss characteristic (i.e., $20 \log[1/M(\omega)]$ versus ω):



(*a*)

- Delay characteristic (i.e., group delay versus ω):



(*b*)

Modified Invariant Impulse-Response Method

- Aliasing errors tend to restrict the application of the invariant impulse-response method to the design of *allpole* filters, i.e., filters that have no zeros in the finite s plane.

Modified Invariant Impulse-Response Method

- Aliasing errors tend to restrict the application of the invariant impulse-response method to the design of *allpole* filters, i.e., filters that have no zeros in the finite s plane.
- However, a *modified* version of the method is available, which can be applied to filters that also have zeros in the finite s plane.

Modified Invariant Impulse-Response Method *Cont'd*

- Consider the transfer function

$$H_A(s) = \frac{H_0 N(s)}{D(s)} = \frac{H_0 \prod_{i=1}^{M}(s - z_i)}{\prod_{i=1}^{N}(s - p_i)}$$

where $N \geq M$.

- Consider the transfer function

$$H_A(s) = \frac{H_0 N(s)}{D(s)} = \frac{H_0 \prod_{i=1}^{M}(s - z_i)}{\prod_{i=1}^{N}(s - p_i)}$$

where $N \geq M$.

- We can write

$$H_A(s) = H_0 \frac{H_{A1}(s)}{H_{A2}(s)}$$

where $H_{A1}(s) = \frac{1}{D(s)}$ and $H_{A2}(s) = \frac{1}{N(s)}$

...

$$H_{A1}(s) = \frac{1}{D(s)} \quad \text{and} \quad H_{A2}(s) = \frac{1}{N(s)}$$

- With M and $N \geq 2$, we have

$$h_A(0+) = \lim_{s \to \infty} [sH_A(s)] = 0$$

...

$$H_{A1}(s) = \frac{1}{D(s)} \quad \text{and} \quad H_{A2}(s) = \frac{1}{N(s)}$$

- With M and $N \geq 2$, we have

$$h_A(0+) = \lim_{s\to\infty} [sH_A(s)] = 0$$

- Also

$$\lim_{s\to\infty} [H_{A1}(s)] \to 0 \quad \text{and} \quad \lim_{s\to\infty} [H_{A2}(s)] \to 0$$

and, consequently,

$$H_{A1}(j\omega) \approx 0 \quad \text{and} \quad H_{A2}(j\omega) \approx 0 \quad \text{for} \;\; |\omega| \geq \frac{\omega_s}{2}$$

for some sufficiently high value of ω_s.

- In effect, by using a sufficiently high sampling frequency, functions $H_{A1}(s)$ and $H_{A2}(s)$ can be considered to be bandlimited analog-filter transfer functions, and for each a discrete-time transfer function can be obtained, as follows, by using the invariant impulse-response method:

$$H_{D1}(z) = \frac{N_1(z)}{D_1(z)} = \sum_{i=1}^{N} \frac{A_i z}{z - e^{Tp_i}} \approx \frac{1}{T} H_{A1}(s) = \frac{1}{T} \frac{1}{D(s)}$$

$$H_{D2}(z) = \frac{N_2(z)}{D_2(z)} = \sum_{i=1}^{M} \frac{B_i z}{z - e^{Tz_i}} \approx \frac{1}{T} H_{A2}(s) = \frac{1}{T} \frac{1}{N(s)}$$

Modified Invariant Impulse-Response Method *Cont'd*

- In effect, by using a sufficiently high sampling frequency, functions $H_{A1}(s)$ and $H_{A2}(s)$ can be considered to be bandlimited analog-filter transfer functions, and for each a discrete-time transfer function can be obtained, as follows, by using the invariant impulse-response method:

$$H_{D1}(z) = \frac{N_1(z)}{D_1(z)} = \sum_{i=1}^{N} \frac{A_i z}{z - e^{Tp_i}} \approx \frac{1}{T} H_{A1}(s) = \frac{1}{T} \frac{1}{D(s)}$$

$$H_{D2}(z) = \frac{N_2(z)}{D_2(z)} = \sum_{i=1}^{M} \frac{B_i z}{z - e^{Tz_i}} \approx \frac{1}{T} H_{A2}(s) = \frac{1}{T} \frac{1}{N(s)}$$

- Therefore, a discrete-time transfer function can be formed as

$$H_D(z) = H_0 \frac{H_{D1}(z)}{H_{D2}(z)} = H_0 \frac{N_1(z) D_2(z)}{N_2(z) D_1(z)} \approx H_0 \frac{N(s)}{D(s)} = H_A(s)$$

. . .

$$H_D(z)H_0\frac{N_1(z)D_2(z)}{N_2(z)D_1(z)} \approx H_A(s)$$

- Evidently, given an arbitrary analog filter with frequency response $H_A(j\omega)$, a corresponding digital filter can be derived with a frequency response

$$H_D(e^{\omega T}) \approx H_A(j\omega) \quad \text{for} \quad |\omega| < \frac{\omega_s}{2}$$

...

$$H_D(z)H_0\frac{N_1(z)D_2(z)}{N_2(z)D_1(z)} \approx H_A(s)$$

- Evidently, given an arbitrary analog filter with frequency response $H_A(j\omega)$, a corresponding digital filter can be derived with a frequency response

$$H_D(e^{\omega T}) \approx H_A(j\omega) \quad \text{for} \quad |\omega| < \frac{\omega_s}{2}$$

- In other words, the modified invariant impulse-response method eliminates the *aliasing problem* associated with the standard invariant impulse-response method.

...

$$H_D(z)H_0\frac{N_1(z)D_2(z)}{N_2(z)D_1(z)} \approx H_A(s)$$

- Evidently, given an arbitrary analog filter with frequency response $H_A(j\omega)$, a corresponding digital filter can be derived with a frequency response

$$H_D(e^{\omega T}) \approx H_A(j\omega) \quad \text{for} \quad |\omega| < \frac{\omega_s}{2}$$

- In other words, the modified invariant impulse-response method eliminates the *aliasing problem* associated with the standard invariant impulse-response method.
- Unfortunately, it also introduces two other problems.

...

$$H_D(z) = H_0 \frac{N_1(z)D_2(z)}{N_2(z)D_1(z)} \approx H_A(s)$$

- Polynomials $N_1(z)$ and $D_1(z)$ are of degree N which is the denominator degree in $H_A(s)$ and polynomials $N_2(z)$ and $D_2(z)$ are of degree M which is the numerator degree in $H_A(s)$.

. . .

$$H_D(z) = H_0 \frac{N_1(z) D_2(z)}{N_2(z) D_1(z)} \approx H_A(s)$$

- Polynomials $N_1(z)$ and $D_1(z)$ are of degree N which is the denominator degree in $H_A(s)$ and polynomials $N_2(z)$ and $D_2(z)$ are of degree M which is the numerator degree in $H_A(s)$.
- Therefore, the order of the derived digital filter is $M + N$ instead of N in the case of the standard invariant impulse-response method.

...

$$H_D(z) = H_0 \frac{N_1(z)D_2(z)}{N_2(z)D_1(z)} \approx H_A(s)$$

- Polynomials $N_1(z)$ and $D_1(z)$ are of degree N which is the denominator degree in $H_A(s)$ and polynomials $N_2(z)$ and $D_2(z)$ are of degree M which is the numerator degree in $H_A(s)$.
- Therefore, the order of the derived digital filter is $M + N$ instead of N in the case of the standard invariant impulse-response method.
- The zeros of $D_1(z)$ are located in the unit circle $|z| = 1$.

. . .

$$H_D(z) = H_0 \frac{N_1(z)D_2(z)}{N_2(z)D_1(z)} \approx H_A(s)$$

- Polynomials $N_1(z)$ and $D_1(z)$ are of degree N which is the denominator degree in $H_A(s)$ and polynomials $N_2(z)$ and $D_2(z)$ are of degree M which is the numerator degree in $H_A(s)$.
- Therefore, the order of the derived digital filter is $M + N$ instead of N in the case of the standard invariant impulse-response method.
- The zeros of $D_1(z)$ are located in the unit circle $|z| = 1$.

  However, the zeros of $N_2(z)$ may be located outside the unit circle, which would render the derived digital filter *unstable*.

Stability Problem

- The stability problem mentioned in the previous slide can be easily circumvented without changing the amplitude response of the filter.

Stability Problem

- The stability problem mentioned in the previous slide can be easily circumvented without changing the amplitude response of the filter.
- Consider a discrete-time transfer function

$$H_D(z) = H_0 \frac{N(z)}{D(z)}$$

and assume that it has poles $p_1, p_2, \ldots, p_K$ that are located outside the unit circle $|z| = 1$.

Stability Problem

- The stability problem mentioned in the previous slide can be easily circumvented without changing the amplitude response of the filter.
- Consider a discrete-time transfer function

$$H_D(z) = H_0 \frac{N(z)}{D(z)}$$

and assume that it has poles $p_1, p_2, \ldots, p_K$ that are located outside the unit circle $|z| = 1$.

$H_D(z)$ can be *stabilized* by simply replacing poles $p_1, p_2, \ldots, p_K$ by their reciprocals $1/p_1, 1/p_2, \ldots, 1/p_K$ and then replacing the multiplier constant H_0 by $H_0 / \prod_{i=1}^{K} p_i$.

(See Chap. 11 for proof.)

Features

- Unlike the standard invariant impulse-response method, the modified method yields *excellent results for elliptic filters*.

Features

- Unlike the standard invariant impulse-response method, the modified method yields *excellent results for elliptic filters*.
- The increased filter order leads to *more complicated realizations* which would be more expensive to implement.

Features

- Unlike the standard invariant impulse-response method, the modified method yields *excellent results for elliptic filters*.
- The increased filter order leads to *more complicated realizations* which would be more expensive to implement.
- Because of the higher order, a software implementation of the discrete-time transfer function obtained would require *more computation*.

Features

- Unlike the standard invariant impulse-response method, the modified method yields *excellent results for elliptic filters*.
- The increased filter order leads to *more complicated realizations* which would be more expensive to implement.
- Because of the higher order, a software implementation of the discrete-time transfer function obtained would require *more computation*.
- The stabilization technique *preserves the amplitude response*, as stated.

Features

- Unlike the standard invariant impulse-response method, the modified method yields *excellent results for elliptic filters*.
- The increased filter order leads to *more complicated realizations* which would be more expensive to implement.
- Because of the higher order, a software implementation of the discrete-time transfer function obtained would require *more computation*.
- The stabilization technique *preserves the amplitude response*, as stated.

  However, the *phase response is changed* which could be a problem in certain applications.

Features

- Unlike the standard invariant impulse-response method, the modified method yields *excellent results for elliptic filters*.
- The increased filter order leads to *more complicated realizations* which would be more expensive to implement.
- Because of the higher order, a software implementation of the discrete-time transfer function obtained would require *more computation*.
- The stabilization technique *preserves the amplitude response*, as stated.

  However, the *phase response is changed* which could be a problem in certain applications.
- The method *provides a theoretical basis* for the matched-z transformation method to be described later.

Example

The transfer function

$$H_A(s) = H_0 \prod_{j=1}^{3} \frac{a_{0j} + s^2}{b_{0j} + b_{1j}s + s^2}$$

where H_0, a_{0j}, and b_{1j} are given in the table, represents an analog lowpass elliptic filter.

j	a_{0j}	b_{0j}	b_{1j}
1	$1.199341E+1$	$3.581929E-1$	$9.508335E-1$
2	2.000130	$6.860742E-1$	$4.423164E-1$
3	1.302358	$8.633304E-1$	$1.088749E-1$
$H_0 = 6.713267E-3$			

Example *Cont'd*

The specifications of the filter are as follows:

- Passband ripple: 0.1 dB
- Minimum stopband loss: 43.46 dB
- Passband edge: $\sqrt{0.8}$ rad/s
- Stopband edge: $1/\sqrt{0.8}$ rad/s

Design a corresponding digital filter by employing the modified invariant impulse-response method.

Assume a sampling frequency $\omega_s = 7.5$ rad/s.

Example *Cont'd*

The design can be obtained through the following steps:

1. Let

$$H_{A1}(s) = \prod_{j=1}^{3} \frac{1}{b_{0j} + b_{1j}s + s^2} \quad \text{and} \quad H_{A2}(s) = \prod_{j=1}^{3} \frac{1}{a_{0j} + s^2}$$

Example *Cont'd*

The design can be obtained through the following steps:

1. Let
$$H_{A1}(s) = \prod_{j=1}^{3} \frac{1}{b_{0j} + b_{1j}s + s^2} \quad \text{and} \quad H_{A2}(s) = \prod_{j=1}^{3} \frac{1}{a_{0j} + s^2}$$

2. Find the poles and residues of $H_{A1}(s)$ and $H_{A2}(s)$.

Example *Cont'd*

The design can be obtained through the following steps:

1. Let

$$H_{A1}(s) = \prod_{j=1}^{3} \frac{1}{b_{0j} + b_{1j}s + s^2} \quad \text{and} \quad H_{A2}(s) = \prod_{j=1}^{3} \frac{1}{a_{0j} + s^2}$$

2. Find the poles and residues of $H_{A1}(s)$ and $H_{A2}(s)$.

3. Form

$$H_{D1}(z) = \sum_{i=1}^{N} \frac{A_i z}{z - e^{Tp_i}} \quad \text{and} \quad H_{D2}(z) = \sum_{i=1}^{M} \frac{B_i z}{z - e^{Tz_i}}$$

Example *Cont'd*

The design can be obtained through the following steps:

1. Let

$$H_{A1}(s) = \prod_{j=1}^{3} \frac{1}{b_{0j} + b_{1j}s + s^2} \quad \text{and} \quad H_{A2}(s) = \prod_{j=1}^{3} \frac{1}{a_{0j} + s^2}$$

2. Find the poles and residues of $H_{A1}(s)$ and $H_{A2}(s)$.

3. Form

$$H_{D1}(z) = \sum_{i=1}^{N} \frac{A_i z}{z - e^{Tp_i}} \quad \text{and} \quad H_{D2}(z) = \sum_{i=1}^{M} \frac{B_i z}{z - e^{Tz_i}}$$

4. Form

$$H_D(z) = H_0 \frac{H_{D1}(z)}{H_{D2}(z)} = H_0 \frac{N_1(z)D_2(z)}{N_2(z)D_1(z)}$$

Example *Cont'd*

The design can be obtained through the following steps:

1. Let

$$H_{A1}(s) = \prod_{j=1}^{3} \frac{1}{b_{0j} + b_{1j}s + s^2} \quad \text{and} \quad H_{A2}(s) = \prod_{j=1}^{3} \frac{1}{a_{0j} + s^2}$$

2. Find the poles and residues of $H_{A1}(s)$ and $H_{A2}(s)$.

3. Form

$$H_{D1}(z) = \sum_{i=1}^{N} \frac{A_i z}{z - e^{Tp_i}} \quad \text{and} \quad H_{D2}(z) = \sum_{i=1}^{M} \frac{B_i z}{z - e^{Tz_i}}$$

4. Form

$$H_D(z) = H_0 \frac{H_{D1}(z)}{H_{D2}(z)} = H_0 \frac{N_1(z)D_2(z)}{N_2(z)D_1(z)}$$

5. Replace poles $p_1, p_2, \ldots, p_K$ of $H_D(z)$ (zeros of $N_2(z)$) located outside the unit circle by their reciprocals and multiplier constant H_0 by $H_0/\prod_{i=1}^{K} p_i$ in order to stabilize the transfer function.

Example *Cont'd*

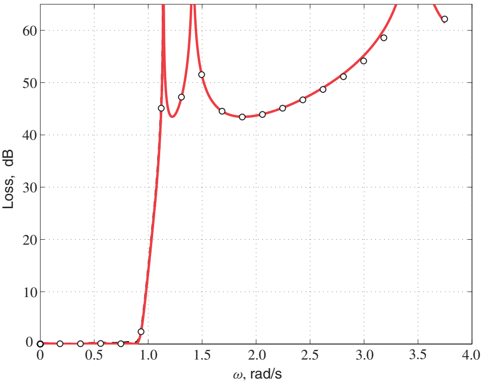
The design procedure gives a transfer function of the form

$$H_D(z) = H_0 \prod_{j=1}^{5} \frac{a_{0j} + a_{1j}z + z^2}{b_{0j} + b_{1j}z + z^2}$$

where H_0, a_{ij}, and b_{ij} are given in the table shown.

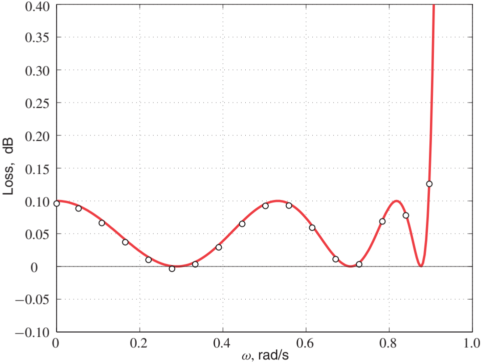
j	a_{0j}	a_{1j}	b_{0j}	b_{1j}
1	1.0	1.942528	4.508735E-1	−1.281134
2	1.0	−7.530225E-1	6.903732E-1	−1.303838
3	1.0	−1.153491	9.128252E-1	−1.362371
4	3.248990E+1	1.955491E+1	5.611278E-2	7.751650E-1
5	1.331746E-2	3.971465E-1	5.611278E-2	7.751650E-1
$H_0 = 3.847141E\text{-}4$				

- Loss characteristic with respect to the baseband: ——— Analog filter; ○ ○ ○ modified impulse-invariant response method.



(*a*)

- Loss characteristic with respect to the passband:
  ——— Analog filter; ∘ ∘ ∘ modified impulse-invariant response method.



(*b*)

Matched-z-Transformation Method

- It was noted early in the history of digital-filter design that the invariant impulse-response method yields a discrete-time transfer function whose poles, $\bar{p}_i$, bear a one-to-one relation to the poles of the continuous-time transfer function, p_i, of the form

$$\bar{p}_i = e^{p_i T}$$

  where T is the sampling period.

Matched-z-Transformation Method

- It was noted early in the history of digital-filter design that the invariant impulse-response method yields a discrete-time transfer function whose poles, $\bar{p}_i$, bear a one-to-one relation to the poles of the continuous-time transfer function, p_i, of the form

$$\bar{p}_i = e^{p_i T}$$

where T is the sampling period.

- It did not take too long for someone to explore calculating the zeros of the discrete-time transfer function, $\bar{z}_i$, from the zeros of the continuous-time transfer function, z_i, using the same relation, i.e.,

$$\bar{z}_i = e^{z_i T}$$

Matched-z-Transformation Method *Cont'd*

- The technique seemed to work well for some types of analog filters but not in others, but it was soon discovered that improved results could be obtained by adding a number of zeros at the Nyquist point.

Matched-z-Transformation Method *Cont'd*

- The technique seemed to work well for some types of analog filters but not in others, but it was soon discovered that improved results could be obtained by adding a number of zeros at the Nyquist point.
- Further heuristic effort identified the number of Nyquist-point zeros needed for the various types of analog filters and the matched-z transformation method was formulated as detailed next.

Matched-z-Transformation Method *Cont'd*

- Given a continuous-time transfer function of the form

$$H_A(s) = H_0 \frac{\prod_{i=1}^{M}(s - z_i)}{\prod_{i=1}^{N}(s - p_i)}$$

a discrete-time transfer function can be obtained as

$$H_D(z) = H_0(z+1)^L \frac{\prod_{i=1}^{M}(z - e^{z_i T})}{\prod_{i=1}^{N}(z - e^{p_i T})}$$

where L is an integer.

Matched-z-Transformation Method *Cont'd*

- The value of L depends on the type of filter and it is given by the table shown.

Type of Filter	LP	HP	BP	BS
Butterworth	N	0	$N/2$	0
Chebyshev	N	0	$N/2$	0
Inverse-Chebyshev, N odd	1	0	n/a	n/a
N even	0	0	1 for odd $N/2$ 0 for even $N/2$	0
Elliptic, N odd	1	0	n/a	n/a
N even	0	0	1 for odd $N/2$ 0 for even $N/2$	0

- If we now compare the discrete-time transfer function given by the modified invariant impulse-response method, i.e.,

$$H_D(z) = H_0 \frac{N_1(z)}{N_2(z)} \cdot \frac{\prod_{i=1}^{M}(z - e^{z_i T})}{\prod_{i=1}^{N}(z - e^{p_i T})}$$

with that obtained by using the matched-z transformation method, i.e.,

$$H_D(z) = H_0(z+1)^L \cdot \frac{\prod_{i=1}^{M}(z - e^{z_i T})}{\prod_{i=1}^{N}(z - e^{p_i T})}$$

we note that the only difference is that the ratio of polynomial $N_1(z)/N_2(z)$ is replaced by the polynomial $(z+1)^L$.

Matched-z-Transformation Method *Cont'd*

- For the classical types of filters (elliptic and inverse-Chebyshev filters), it turns out that $N1(z)$ and $N_2(z)$ are mirror image polynomials with zeros on the negative real axis of the z plane clustered near the Nyquist point and, consequently,

$$\frac{N_1(z)}{N_2(z)} \approx (z+1)^L$$

Matched-z-Transformation Method *Cont'd*

- For the classical types of filters (elliptic and inverse-Chebyshev filters), it turns out that $N1(z)$ and $N_2(z)$ are mirror image polynomials with zeros on the negative real axis of the z plane clustered near the Nyquist point and, consequently,

$$\frac{N_1(z)}{N_2(z)} \approx (z+1)^L$$

- In effect, at least, for classical filters, the discrete-time transfer function obtained with the matched-z method is an *approximation* of that obtained with the modified invariant impulse-response method.

Advantages

- *Simple to apply* — the design can be done with a calculator.

Advantages

- *Simple to apply* — the design can be done with a calculator.
- *Works moderately well* not only *for lowpass and bandpass filters* but also for *highpass and bandstop filters* including *elliptic filters*, i.e., no aliasing problems.

Advantages

- *Simple to apply* — the design can be done with a calculator.
- *Works moderately well* not only *for lowpass and bandpass filters* but also for *highpass and bandstop filters* including *elliptic filters*, i.e., no aliasing problems.
- The absence of $N_2(z)$ *eliminates the stability problem* associated with the modified invariant impulse-response method.

Disadvantages

- The *passband loss characteristic* of the digital filter is *seriously distorted* relative to that of the analog filter.

Disadvantages

- The *passband loss characteristic* of the digital filter is *seriously distorted* relative to that of the analog filter.
- A *high sampling frequency is usually necessary* to achieve good results which can introduce certain problems.

  A high sampling frequency corresponds to a reduced sampling period and, therefore, the amount of processing that can be done between samples is reduced.

Disadvantages

- The *passband loss characteristic* of the digital filter is *seriously distorted* relative to that of the analog filter.
- A *high sampling frequency is usually necessary* to achieve good results which can introduce certain problems.

  A high sampling frequency corresponds to a reduced sampling period and, therefore, the amount of processing that can be done between samples is reduced.
- The *multiplier constant needs to be adjusted* at the end of the design (see Chap. 11) for details).

Example

- The matched-z transformation method was used to redesign the elliptic filter considered earlier and the design obtained is as follows:

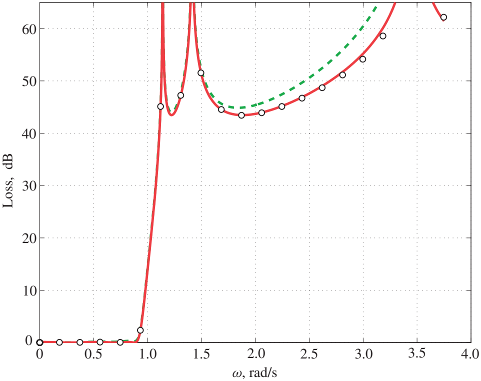
$$H_D(z) = H_0 \prod_{j=1}^{5} \frac{a_{0j} + a_{1j}z + z^2}{b_{0j} + b_{1j}z + z^2}$$

where H_0, a_{ij}, and b_{ij} are given in the table shown.

j	a_{0j}	a_{1j}	b_{0j}	b_{1j}
1	1.0	-1.153491	$9.128252E$-1	-1.362371
2	$3.248990E+1$	$1.955491E+1$	$5.611278E$-2	$7.751650E$-1
3	$1.331746E$-2	$3.971465E$-1	$5.611278E$-2	$7.751650E$-1
$H_0 = 3.847141E$-4				

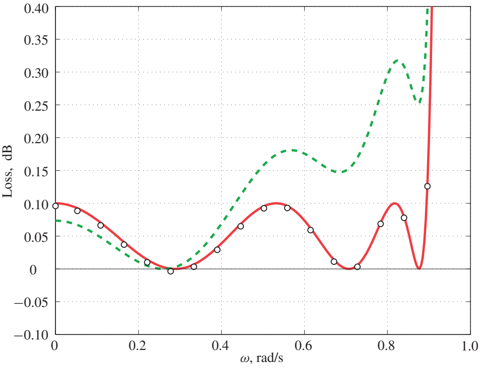
Example *Cont'd*

- Loss characteristics with respect to the baseband:
  ——— Analog filter; ○ ○ ○ modified impulse-invariant response method; - - - - - matched-z transformation method.



(*a*)

- Loss characteristics with respect to the passband:
  ——— Analog filter; ∘ ∘ ∘ modified impulse-invariant response method; - - - - - matched-z transformation method.



(*b*)

This slide concludes the presentation.

Thank you for your attention.