Chapter 12
RECURSIVE (IIR) FILTERS SATISFYING PRESCRIBED SPECIFICATIONS

12.1 Introduction
12.2 Design Procedure
12.3 Design Formulas
12.4 Design Using the Formulas and Tables

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July 14, 2018

Introduction

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  - The maximum passband loss and minimum stopband loss in the analog filter are preserved in the digital filter.
  - Given a stable causal analog filter, a stable causal digital filter is obtained.
- Due to these important advantages, the bilinear transformation method is used quite extensively in practice.

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- As was demonstrated in the previous presentation, the main problem with the method is the so-called *warping effect* which introduces frequency-scale distortion.

- If $\omega_1, \omega_2, \ldots, \omega_K$ are the passband and stopband edges in the analog filter, then the corresponding passband and stopband edges in the derived digital filter are given by

$$\Omega_i = \frac{2}{T} \tan^{-1} \frac{\omega_i T}{2} \quad \text{for} \quad i = 1, 2, \ldots, K$$

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- If prescribed passband and stopband edges $\tilde{\Omega}_1, \tilde{\Omega}_2, \ldots, \tilde{\Omega}_K$ are to be achieved, then the analog filter *must be prewarped* before the application of the bilinear transformation to ensure that its band edges are given by

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- If this could be accomplished, then the band edges of the digital filter would assume their prescribed values Ω_i since

$$\Omega_i = \frac{2}{T} \tan^{-1} \frac{\omega_i T}{2} = \frac{2}{T} \tan^{-1} \left(\frac{T}{2} \cdot \frac{2}{T} \tan \frac{\tilde{\Omega}_i T}{2} \right) = \tilde{\Omega}_i$$

Introduction *Cont'd*

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- This prewarping procedure amounts to increasing the actual band edges of the analog filter by appropriate amounts so that the effects of warping will cause the band edges to assume their specified values.
- The procedure can be used to design lowpass (LP), highpass (HP), bandpass (BP), and bandstop (BS) digital filters that would satisfy arbitrary prescribed specifications.
- It is applicable to *all the classical families* of analog filters, namely, Butterworth, Chebyshev, Inverse-Chebyshev, and elliptic filters.

Classical Analog Filters

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- The transfer functions of the classical analog-filter families are usually reported in the literature in *normalized* LP form whereby the passband edge is typically of order of unity.
- This is because *denormalized* transfer functions for LP, HP, BP, and BS filters of arbitrary cutoff frequencies can be readily obtained using a family of *analog-filter transformations*.

Design Procedure

- Consider a normalized analog LP filter characterized by $H_N(s)$ with a loss

$$A_N(\omega) = 20\log\frac{1}{|H_N(j\omega)|} = 20\log\frac{1}{M(\omega)}$$

and assume that

$$0 \le A_N(\omega) \le A_p \quad \text{for} \quad 0 \le |\omega| \le \omega_p$$
$$A_N(\omega) \ge A_a \quad \text{for} \quad \omega_a \le |\omega| \le \infty$$

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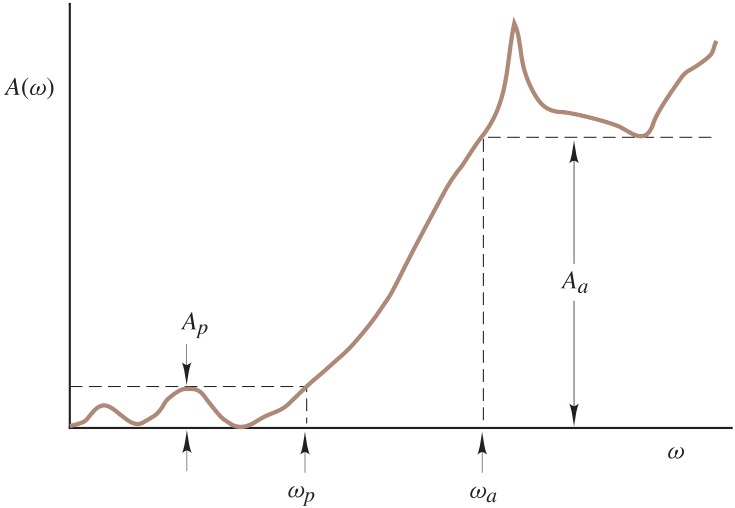
  Notes:

  - Recall that $M(\omega) = |H(j\omega)|$ is the gain in an analog system.
  - The loss $A_N(\omega)$ is sometimes referred to as attenuation.

Design Procedure *Cont'd*



Design Procedure *Cont'd*

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- A denormalized analog LP, HP, BP, or BS filter that has the same passband ripple and minimum stopband loss as a given normalized LP filter can be derived from the normalized LP filter through the following general steps:

  1. Apply the transformation $s = f_X(\bar{s})$

  $$H_X(\bar{s}) = H_N(s)\Big|_{s=f_X(\bar{s})}$$

  where $f_X(\bar{s})$ is one of the four standard analog-filter transformations shown in the next slide.

Standard forms of $f_X(\bar{s})$

Type	$f_X(\bar{s})$
LP to LP	$\lambda\bar{s}$
LP to HP	$\lambda/\bar{s}$
LP to BP	$\frac{1}{B}\left(\bar{s}+\dfrac{\omega_0^2}{\bar{s}}\right)$
LP to BS	$\dfrac{B\bar{s}}{\bar{s}^2+\omega_0^2}$

2. Apply the bilinear transformation to $H_X(\bar{s})$, i.e.,

$$H_D(z) = H_X(\bar{s})\Big|_{\bar{s}=\frac{2}{T}\left(\frac{z-1}{z+1}\right)}$$

- The digital filter designed by this method will have the required passband and stopband edges *only if the parameters* λ, ω_0, *and* B of the analog-filter transformations *and the order* of the continuous-time normalized LP transfer function, $H_N(s)$, are chosen appropriately.

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- This is obviously a difficult problem but general solutions are available for LP, HP, BP, and BS filters of the Butterworth, Chebyshev, inverse-Chebyshev, and elliptic filter families.

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- The derivation of the required formulas for LP filters is examined next.

Design Formulas for LP Filters

- *Step 1:* Consider a normalized LP filter with a loss

$$A_N(\omega) = 20 \log \frac{1}{|H_N(j\omega)|}$$

and assume that

$$0 \le A_N(\omega) \le A_p \qquad \text{for} \ \ 0 \le |\omega| \le \omega_p$$

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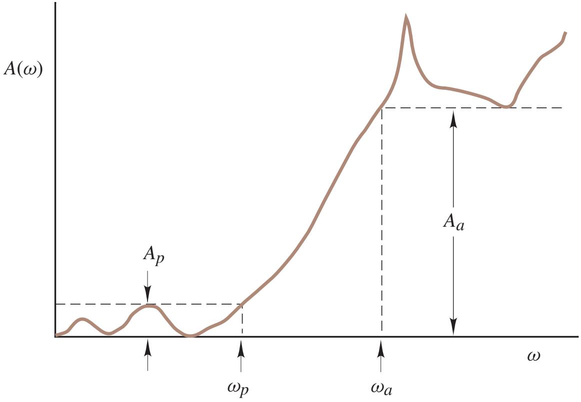
Design Formulas for LP Filters *Cont'd*



- *Step 2:* A *denormalized* LP analog filter can be obtained by applying the LP-to-LP transformation

$$H_{LP}(\bar{s}) = H_N(s)\Big|_{s=\lambda\bar{s}}$$

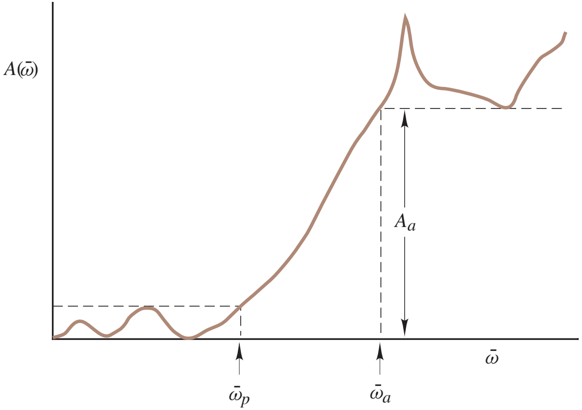
Design Formulas for LP Filters *Cont'd*

- *Step 2:* A *denormalized* LP analog filter can be obtained by applying the LP-to-LP transformation

$$H_{LP}(\bar{s}) = H_N(s)\Big|_{s=\lambda\bar{s}}$$

- The LP-to-LP transformation will scale the passband and stopband edges of the normalized LP filter ω_p and ω_a to $\bar{\omega}_p$ and $\bar{\omega}_a$ as shown in the next slide.

Design Formulas for LP Filters *Cont'd*



...

$$H_{LP}(\bar{s}) = H_N(s)\Big|_{s=\lambda\bar{s}}$$

- If we let $s = j\omega$ and $\bar{s} = j\bar{\omega}$, we get

$$|H_{LP}(j\bar{\omega})| = |H_N(j\omega)|$$

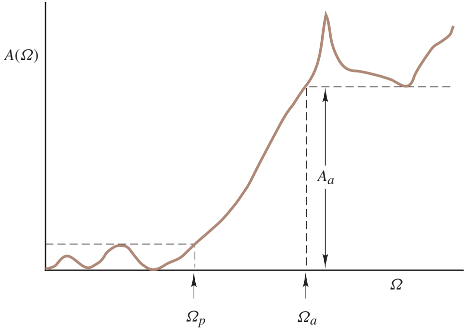
provided that $\omega = \lambda\bar{\omega}$

Hence

$$\omega_p = \lambda\bar{\omega}_p \quad \text{and} \quad \omega_a = \lambda\bar{\omega}_a$$

where $\bar{\omega}_p$ and $\bar{\omega}_a$ denote the passband and stopband edges, respectively, in the *denormalized* LP filter.

- *Step 3:* The bilinear transformation will scale the passband and stopband edges of the denormalized LP filter $\bar{\omega}_p$ and $\bar{\omega}_a$ to Ω_p and Ω_a as shown:



- Applying the bilinear transformation to the denormalized LP filter, we get

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- If we now let $z = e^{j\Omega T}$ and $\bar{s} = j\bar{\omega}$, we get

$$|H_D(e^{j\Omega T})| = |H_{LP}(j\bar{\omega})|$$

provided that
$$\bar{\omega} = \frac{2}{T}\tan\frac{\Omega T}{2}$$

Thus

$$\bar{\omega}_p = \frac{2}{T}\tan\frac{\Omega_p T}{2} \quad \text{and} \quad \bar{\omega}_a = \frac{2}{T}\tan\frac{\Omega_a T}{2}$$

. . .

$$\omega_p = \lambda\bar{\omega}_p, \quad \omega_a = \lambda\bar{\omega}_a$$
$$\bar{\omega}_p = \frac{2}{T}\tan\frac{\Omega_p T}{2}, \quad \bar{\omega}_a = \frac{2}{T}\tan\frac{\Omega_a T}{2}$$

- *Step 4:* From these formulas, we get

$$\omega_p = \frac{2}{T}\lambda\tan\frac{\Omega_p T}{2} \quad \text{and} \quad \omega_a = \frac{2}{T}\lambda\tan\frac{\Omega_a T}{2}$$

Therefore,

$$\lambda = \frac{T\omega_p}{2\tan(\Omega_p T/2)} \quad \text{and} \quad \frac{\omega_p}{\omega_a} = \frac{\tan(\Omega_p T/2)}{\tan(\Omega_a T/2)}$$

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- We note that if ω_p, Ω_p, Ω_a, and $\omega_s = 2\pi/T$ are known, then

Design Formulas for LP Filters *Cont'd*

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  - the parameter of the LP-to-LP transformation, λ, and
  - the ratio ω_p/ω_a can be determined.
- The ratio ω_p/ω_a is commonly referred to as the *selectivity* of the filter and it can be used to determine the *minimum* filter order that would satisfy the required specifications.

- As may be recalled, the analog-filter transformations and the bilinear transformation preserve the maximum passband loss, A_p, and the minimum stopband loss, A_a, of the normalized LP filter.

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  Thus the derived digital filter will automatically satisfy these specifications.

- *Step 5:* Let us assume that the specified passband and stopband edges are $\tilde{\Omega}_p$ and $\tilde{\Omega}_a$, respectively.

Design Formulas for LP Filters *Cont'd*

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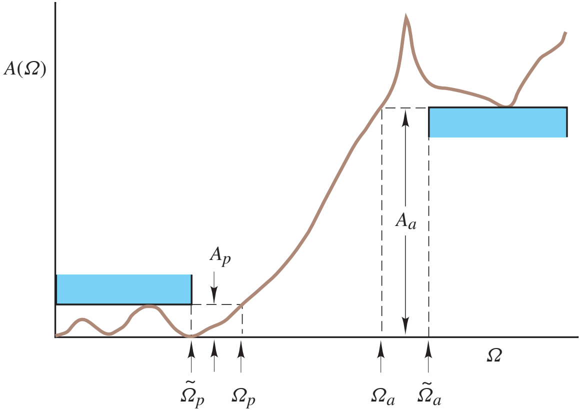
- The *next best thing* would be to design the filter such that

$$\Omega_p \geq \tilde{\Omega}_p \quad \text{and} \quad \Omega_a \leq \tilde{\Omega}_a$$

Design Formulas for LP Filters *Cont'd*

- Obviously, an infinite number of choices are available.



- One possibility would be to satisfy the specs *exactly* at the passband edge and *oversatisfy* the specs at the stopband edge, so we can assign

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Design Formulas for LP Filters *Cont'd*

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- From a previous slide,

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$$\lambda = \frac{T\omega_p}{2\tan(\tilde{\Omega}_p T/2)}$$

$$\frac{\omega_p}{\omega_a} = \frac{\tan(\Omega_p T/2)}{\tan(\Omega_a T/2)} \geq \frac{\tan(\tilde{\Omega}_p T/2)}{\tan(\tilde{\Omega}_a T/2)} = K_0$$

. . .

$$\lambda = \frac{T\omega_p}{2\tan(\tilde{\Omega}_p T/2)}$$

$$\frac{\omega_p}{\omega_a} \geq K_0 \quad \text{where} \quad K_0 = \frac{\tan(\tilde{\Omega}_p T/2)}{\tan(\tilde{\Omega}_a T/2)}$$

- Summarizing, if the specifications and the sampling frequency are known, the selectivity of the normalized LP filter can be determined.

Design Formulas for LP Filters *Cont'd*

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- Summarizing, if the specifications and the sampling frequency are known, the selectivity of the normalized LP filter can be determined.
- Once the approximation type, e.g., Butterworth, Chebyshev, etc., is chosen the passband edge ω_p and the minimum filter order to achieve a maximum passband loss, A_p, and a minimum stopband loss, A_a, can be determined as will be shown later.

  After that, the parameter of the LP-to-LP transformation, λ, can be evaluated.

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  After that, the parameter of the LP-to-LP transformation, λ, can be evaluated.
- This approach can be readily extended to *highpass filters*.

Formulas for LP and HP Filters

LP	$\dfrac{\omega_p}{\omega_a} \geq K_0$
	$\lambda = \dfrac{\omega_p T}{2\tan(\tilde{\Omega}_p T/2)}$
HP	$\dfrac{\omega_p}{\omega_a} \geq \dfrac{1}{K_0}$
	$\lambda = \dfrac{2\omega_p \tan(\tilde{\Omega}_p T/2)}{T}$
where	$K_0 = \dfrac{\tan(\tilde{\Omega}_p T/2)}{\tan(\tilde{\Omega}_a T/2)}$

Design Formulas for BP Filters

An outline of the procedure for the derivation of general solutions for BP filters is as follows:

1. Assume that a continuous-time normalized LP transfer function, $H_N(s)$, is available that would give the required passband ripple, A_p, and minimum stopband loss, A_a.

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An outline of the procedure for the derivation of general solutions for BP filters is as follows:

1. Assume that a continuous-time normalized LP transfer function, $H_N(s)$, is available that would give the required passband ripple, A_p, and minimum stopband loss, A_a.

   Let the passband and stopband edges of the analog filter be ω_p and ω_a, respectively.

2. Apply the LP-to-BP analog-filter transformation to $H_N(s)$ to obtain a denormalized continuous-time transfer function $H_{BP}(\bar{s})$ and find the lower and upper passband edges, $\bar{\omega}_{p1}$ and $\bar{\omega}_{p2}$, and lower and upper stopband edges, $\bar{\omega}_{a1}$ and $\bar{\omega}_{a2}$, of the denormalized bandpass filter.

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3. Apply the bilinear transformation to $H_{BP}(\bar{s})$ to obtain a discrete-time transfer function $H_D(z)$ and find the lower and upper passband edges, Ω_{p1} and Ω_{p2}, and lower and upper stopband edges, Ω_{a1} and Ω_{a2}, of the digital filter obtained.

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4. Obtain formulas for the parameters of the LP-to-BP transformation, ω_0 and B, and the selectivity of the normalized LP filter, ω_p/ω_a.

5. At this point, assume that the derived discrete-time transfer function has passband and stopband edges that satisfy the relations

$$\Omega_{p1} \le \tilde{\Omega}_{p1} \quad \text{and} \quad \Omega_{p2} \ge \tilde{\Omega}_{p2}$$

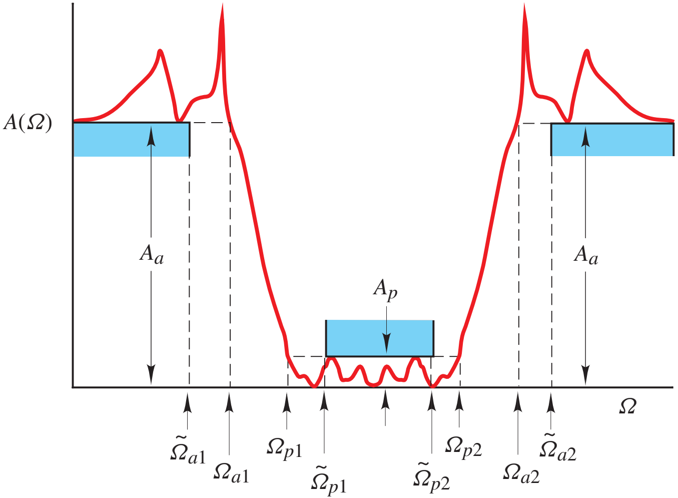
and

$$\Omega_{a1} \ge \tilde{\Omega}_{a1} \quad \text{and} \quad \Omega_{a2} \le \tilde{\Omega}_{a2}$$

where

- Ω_{p1} and Ω_{p2} are the actual lower and upper passband edges,
- $\tilde{\Omega}_{p1}$ and $\tilde{\Omega}_{p2}$ are the *prescribed* lower and upper passband edges,
- Ω_{a1} and Ω_{a2} are the actual lower and upper stopband edges,
- $\tilde{\Omega}_{a1}$ and $\tilde{\Omega}_{a2}$ are the *prescribed* lower and upper stopband edges, respectively.

Design Formulas for BP Filters *Cont'd*



6. Assign

$$\Omega_{p1} = \tilde{\Omega}_{p1} \quad \text{and} \quad \Omega_{p2} = \tilde{\Omega}_{p2}$$

and

$$\Omega_{a1} \geq \tilde{\Omega}_{a1} \quad \text{and} \quad \Omega_{a2} \leq \tilde{\Omega}_{a2}$$

and obtain formulas for the parameters of the LP-to-BP transformation, ω_0 and B, and the selectivity of the normalized LP filter, ω_p/ω_a, in terms of the specified lower and upper passband edges, Ω_{p1} and Ω_{p2}, and the lower and upper stopband edges, Ω_{a1} and Ω_{a2}.

Design Formulas for BP Filters *Cont'd*

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8. The application of this procedure yields the formulas summarized in the next two slides.

   (See textbook for the derivation of the design formulas.)

Design Formulas for BP Filters *Cont'd*

BP	$\omega_0 = \dfrac{2\sqrt{K_B}}{T}$
	$\dfrac{\omega_p}{\omega_a} \geq \begin{cases} K_1 & \text{if } K_C \geq K_B \\ K_2 & \text{if } K_C < K_B \end{cases}$
	$B = \dfrac{2K_A}{T\omega_p}$

where

$$K_A = \tan\frac{\tilde{\Omega}_{p2}T}{2} - \tan\frac{\tilde{\Omega}_{p1}T}{2} \qquad K_B = \tan\frac{\tilde{\Omega}_{p1}T}{2}\tan\frac{\tilde{\Omega}_{p2}T}{2}$$

$$K_C = \tan\frac{\tilde{\Omega}_{a1}T}{2}\tan\frac{\tilde{\Omega}_{a2}T}{2} \qquad K_1 = \frac{K_A\tan(\tilde{\Omega}_{a1}T/2)}{K_B - \tan^2(\tilde{\Omega}_{a1}T/2)}$$

$$K_2 = \frac{K_A\tan(\tilde{\Omega}_{a2}T/2)}{\tan^2(\tilde{\Omega}_{a2}T/2) - K_B}$$

Design Formulas for BS Filters

BS	$\omega_0 = \dfrac{2\sqrt{K_B}}{T}$
	$\dfrac{\omega_p}{\omega_a} \geq \begin{cases} \dfrac{1}{K_2} & \text{if } K_C \geq K_B \\ \dfrac{1}{K_1} & \text{if } K_C < K_B \end{cases}$
	$B = \dfrac{2K_A\omega_p}{T}$

where

$$K_A = \tan\frac{\tilde{\Omega}_{p2}T}{2} - \tan\frac{\tilde{\Omega}_{p1}T}{2} \qquad K_B = \tan\frac{\tilde{\Omega}_{p1}T}{2}\tan\frac{\tilde{\Omega}_{p2}T}{2}$$

$$K_C = \tan\frac{\tilde{\Omega}_{a1}T}{2}\tan\frac{\tilde{\Omega}_{a2}T}{2} \qquad K_1 = \frac{K_A\tan(\tilde{\Omega}_{a1}T/2)}{K_B - \tan^2(\tilde{\Omega}_{a1}T/2)}$$

$$K_2 = \frac{K_A\tan(\tilde{\Omega}_{a2}T/2)}{\tan^2(\tilde{\Omega}_{a2}T/2) - K_B}$$

Formulas for n and ω_p

- The formulas presented so far apply to any type of normalized analog LP filter with a loss that would satisfy the conditions:

$$0 \le A_N(\omega) \le A_p \quad \text{for} \quad 0 \le |\omega| \le \omega_p$$
$$A_N(\omega) \ge A_a \quad \text{for} \quad \omega_a \le |\omega| \le \infty$$

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- However, the values of the required filter order, n, and the normalized passband edge, ω_p, depend on the analog-filter family, i.e., Butterworth, Chebyshev, inverse-Chebyshev, or elliptic.

- Formulas for these parameters have been obtained in Chap. 10.

Formulas for n and ω_p – Butterworth Filters

- For Butterworth filters, we have

$$n \geq \frac{\log D}{2\log(1/K)} \quad \text{where} \quad D = \frac{10^{0.1A_a} - 1}{10^{0.1A_p} - 1}$$

$$\omega_p = (10^{0.1A_p} - 1)^{1/2n}$$

Parameter K depends on the type of denormalized filter required, i.e, LP, HP, BP, or BS.

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  Parameter K depends on the type of denormalized filter required, i.e, LP, HP, BP, or BS.

- The required formulas for Butterworth filters are summarized in the next slide.

Formulas for Butterworth Filters *Cont'd*

LP	$K = K_0$
HP	$K = \dfrac{1}{K_0}$
BP	$K = \begin{cases} K_1 & \text{if } K_C \geq K_B \\ K_2 & \text{if } K_C < K_B \end{cases}$
BS	$K = \begin{cases} \dfrac{1}{K_2} & \text{if } K_C \geq K_B \\ \dfrac{1}{K_1} & \text{if } K_C < K_B \end{cases}$

$$n \geq \frac{\log D}{2\log(1/K)}, \quad D = \frac{10^{0.1A_a} - 1}{10^{0.1A_p} - 1}$$

$$\omega_p = (10^{0.1A_p} - 1)^{1/2n}$$

Formulas for n and ω_p – Chebyshev Filters

- For Chebyshev filters, we have

$$n \geq \frac{\cosh^{-1}\sqrt{D}}{\cosh^{-1}(1/K)} \quad \text{where} \quad D = \frac{10^{0.1A_a} - 1}{10^{0.1A_p} - 1}$$

$$\omega_p = 1$$

As before, K depends on the type of denormalized filter required.

Formulas for n and ω_p – Chebyshev Filters

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$$\omega_p = 1$$

  As before, K depends on the type of denormalized filter required.

- The required formulas for Chebyshev filters are summarized in the next slide.

Formulas for Chebyshev Filters *Cont'd*

LP	$K = K_0$
HP	$K = \dfrac{1}{K_0}$
BP	$K = \begin{cases} K_1 & \text{if } K_C \geq K_B \\ K_2 & \text{if } K_C < K_B \end{cases}$
BS	$K = \begin{cases} \dfrac{1}{K_2} & \text{if } K_C \geq K_B \\ \dfrac{1}{K_1} & \text{if } K_C < K_B \end{cases}$

$$n \geq \frac{\cosh^{-1}\sqrt{D}}{\cosh^{-1}(1/K)}, \quad D = \frac{10^{0.1A_a} - 1}{10^{0.1A_p} - 1}$$

$$\omega_p = 1$$

Formulas for n and ω_p – Inverse-Chebyshev Filters

- For inverse-Chebyshev filters, we have

$$n \geq \frac{\cosh^{-1}\sqrt{D}}{\cosh^{-1}(1/K)} \quad \text{where} \quad D = \frac{10^{0.1A_a} - 1}{10^{0.1A_p} - 1}$$

$$\omega_p = \frac{1}{\cosh\left(\frac{1}{n}\cosh^{-1}\sqrt{D}\right)}$$

Formulas for n and ω_p – Inverse-Chebyshev Filters

- For inverse-Chebyshev filters, we have

$$n \geq \frac{\cosh^{-1}\sqrt{D}}{\cosh^{-1}(1/K)} \quad \text{where} \quad D = \frac{10^{0.1A_a} - 1}{10^{0.1A_p} - 1}$$

$$\omega_p = \frac{1}{\cosh\left(\frac{1}{n}\cosh^{-1}\sqrt{D}\right)}$$

- The required formulas for inverse-Chebyshev filters are summarized in the next slide.

Formulas for Inverse-Chebyshev Filters *Cont'd*

LP	$K = K_0$
HP	$K = \dfrac{1}{K_0}$
BP	$K = \begin{cases} K_1 & \text{if } K_C \geq K_B \\ K_2 & \text{if } K_C < K_B \end{cases}$
BS	$K = \begin{cases} \dfrac{1}{K_2} & \text{if } K_C \geq K_B \\ \dfrac{1}{K_1} & \text{if } K_C < K_B \end{cases}$

$$n \geq \frac{\cosh^{-1}\sqrt{D}}{\cosh^{-1}(1/K)}, \quad D = \frac{10^{0.1A_a} - 1}{10^{0.1A_p} - 1}$$

$$\omega_p = \frac{1}{\cosh\left(\frac{1}{n}\cosh^{-1}\sqrt{D}\right)}$$

Formulas for n and ω_p – Elliptic Filters

- For elliptic filters, the required filter order is given by

$$n \geq \frac{\log 16D}{\log 1/q} \quad \text{where} \quad D = \frac{10^{0.1A_a} - 1}{10^{0.1A_p} - 1}$$

  In elliptic filters, an *arbitrary* selectivity k in the range $0 < k < 1$ can be chosen and, therefore, we can select

$$k = K$$

- Therefore, from Chap. 10

$$\omega_p = \sqrt{k}$$

- The required formulas for elliptic filters are summarized in the next slide.

Formulas for Elliptic Filters *Cont'd*

	k	ω_p
LP	$K = K_0$	$\sqrt{K_0}$
HP	$K = \dfrac{1}{K_0}$	$\dfrac{1}{\sqrt{K_0}}$
BP	$K = \begin{cases} K_1 & \text{if } K_C \geq K_B \\ K_2 & \text{if } K_C < K_B \end{cases}$	$\sqrt{K_1}$ $\sqrt{K_2}$
BS	$K = \begin{cases} \dfrac{1}{K_2} & \text{if } K_C \geq K_B \\ \dfrac{1}{K_1} & \text{if } K_C < K_B \end{cases}$	$\dfrac{1}{\sqrt{K_2}}$ $\dfrac{1}{\sqrt{K_1}}$

$$n \geq \frac{\log 16D}{\log 1/q} \quad \text{where} \quad D = \frac{10^{0.1A_a} - 1}{10^{0.1A_p} - 1}$$

Design Procedure

A digital LP, HP, BP, or BS filter that would satisfy prescribed specifications, i.e., maximum passband loss A_p, minimum stopband loss, A_a, and specified passband and stopband edges, $\tilde{\Omega}_p$ and $\tilde{\Omega}_a$ or $\tilde{\Omega}_{p1}$, $\tilde{\Omega}_{p2}$ and $\tilde{\Omega}_{a1}$, $\tilde{\Omega}_{a2}$, can be designed through the following steps:

1. Determine n and ω_p, and for elliptic filters also k, using the formulas in the tables presented (see Tables 12.4 to 12.6 in the textbook).
2. Determine λ for LP and HP filters (see Table 12.2) or B and ω_0 for BP and BS filters using the formulas in the tables presented (see Tables 12.2 and 12.3).
3. Form the normalized LP transfer function (see Chap. 10).
4. Apply the appropriate analog-filter transformation.
5. Apply the bilinear transformation.

Example – HP Filter

An HP filter that would satisfy the following specifications is required:

$$A_p = 1 \text{ dB}, \quad A_a = 45 \text{ dB}, \quad \tilde{\Omega}_p = 3.5 \text{ rad/s},$$

$$\tilde{\Omega}_a = 1.5 \text{ rad/s}, \quad \omega_s = 10 \text{ rad/s}.$$

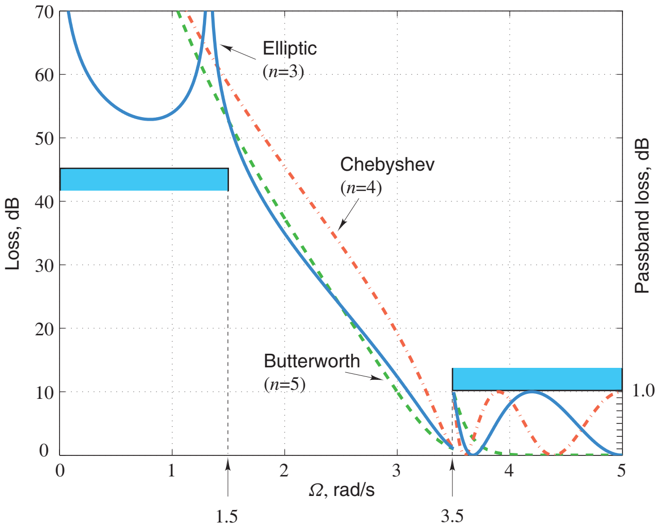
Design a Butterworth, a Chebyshev, and then an elliptic digital filter.

Example – HP Filter *Cont'd*

Solution

Filter type	n	ω_p	λ
Butterworth	5	0.873610	5.457600
Chebyshev	4	1.0	6.247183
elliptic	3	0.509526	3.183099

Example – HP Filter *Cont'd*



Design an elliptic BP filter that would satisfy the following specifications:

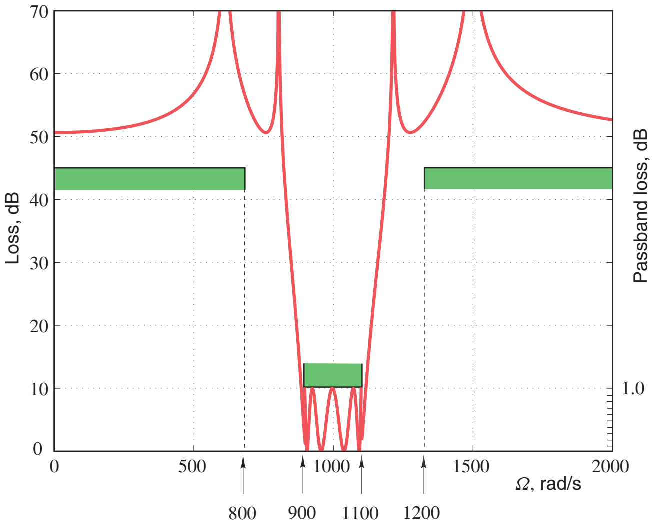
$$A_p = 1 \text{ dB}, \quad A_a = 45 \text{ dB}, \quad \tilde{\Omega}_{p1} = 900 \text{ rad/s}, \quad \tilde{\Omega}_{p2} = 1100 \text{ rad/s}$$
$$\tilde{\Omega}_{a1} = 800 \text{ rad/s}, \quad \tilde{\Omega}_{a2} = 1200 \text{ rad/s}, \quad \omega_s = 6000 \text{ rad/s}$$

Solution

$$k = 0.515957, \quad \omega_p = 0.718302, \quad n = 4$$

$$\omega_0 = 1,098.609, \quad B = 371.9263$$

Example – BP Filter *Cont'd*



Example – BS Filter

Design a Chebyshev BS filter that would satisfy the following specifications:

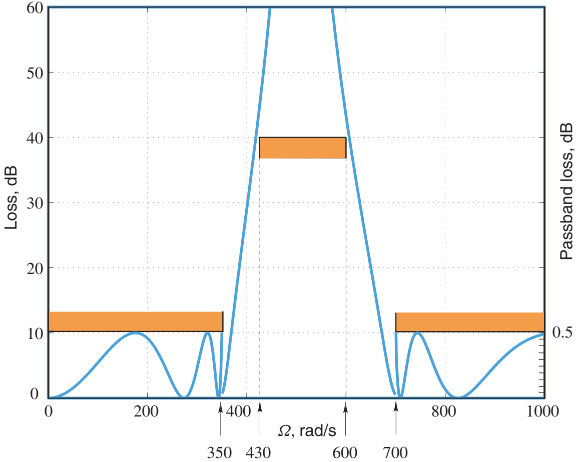
$$A_p = 0.5 \text{ dB}, \quad A_a = 40 \text{ dB}, \quad \tilde{\Omega}_{p1} = 350 \text{ rad/s}, \quad \tilde{\Omega}_{p2} = 700 \text{ rad/s}$$

$$\tilde{\Omega}_{a1} = 430 \text{ rad/s}, \quad \tilde{\Omega}_{a2} = 600 \text{ rad/s}, \quad \omega_s = 3000 \text{ rad/s}$$

Solution

$$\omega_p = 1.0, \quad n = 5, \quad \omega_0 = 561,4083$$

$$B = 493,2594$$


60
50
40
30
20
10
0
Loss, dB
Passband loss, dB
0.5
0
200
400
800
1000
Ω, rad/s
350
430
600
700

D-Filter

A DSP software package that incorporates the design techniques described in this presentation is *D-Filter*.

For more information about D-Filter or to download a *free* copy, click the following link:

http://ece.uvic.ca/~dsp/Software-ne.html

Summary

- An indirect design procedure for recursive filters that leads to a complete description of the transfer function *in closed form* either in terms of its zeros and poles or its coefficients has been described.

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- The procedure requires *very little computation* and leads to very precise *optimal* designs.
- It can be used to design LP, HP, BP, and BS filters for all the classical analog-filter families, namely, Butterworth, Chebyshev, Inverse-Chebyshev, and elliptic.

Summary

- An indirect design procedure for recursive filters that leads to a complete description of the transfer function *in closed form* either in terms of its zeros and poles or its coefficients has been described.
- The procedure requires *very little computation* and leads to very precise *optimal* designs.
- It can be used to design LP, HP, BP, and BS filters for all the classical analog-filter families, namely, Butterworth, Chebyshev, Inverse-Chebyshev, and elliptic.
- All these designs can be carried out by using DSP software package D-Filter.

This slide concludes the presentation.
Thank you for your attention.