Chapter 16
DESIGN OF RECURSIVE FILTERS USING OPTIMIZATION METHODS
16.1 Introduction
16.2 Problem Formulation
16.3 Newton's Method

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Introduction

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- These methods lead to a complete description of the transfer function in closed form, either in terms of its zeros and poles or its coefficients.
- Consequently, they are very efficient and lead to very precise designs.
- Their main disadvantage is that they are applicable only for the design of classical-type filters such as Butterworth lowpass filters, elliptic bandpass filters, etc.

Introduction *Cont'd*

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- A norm of the error function is then minimized with respect to the transfer-function coefficients.

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- A norm of the error function is then minimized with respect to the transfer-function coefficients.

- Like the Remez algorithm described in Chap. 15, optimization methods for the design of recursive filters are *iterative*.

  As a result, they usually involve a large amount of computation.

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  - formulation of an optimization problem
  - fundamentals of optimization
  - the basic *Newton algorithm*

- More sophisticated practical optimization algorithms such as *quasi-Newton* and *minimax* algorithms along with their application for the design of recursive digital filters will be presented later.

Problem Formulation

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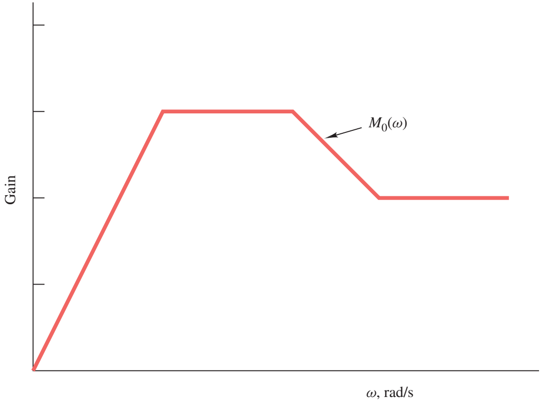
2. Minimize the objective function with respect to the transfer function coefficients.

- Let us assume that we need to design an Nth-order recursive filter with a piecewise-linear amplitude response $M_0(\omega)$ such as that shown in the figure.



Problem Formulation *Cont'd*

- The transfer function of the filter can be expressed as

$$H(z) = H_0 \prod_{j=1}^{J} \frac{a_{0j} + a_{1j}z + z^2}{b_{0j} + b_{1j}z + z^2}$$

where a_{ij} and b_{ij} are real coefficients, $J = N/2$, and H_0 is a positive multiplier constant.

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- As presented, $H(z)$ would be of even order; however, an odd-order $H(z)$ can be obtained by letting

$$a_{0j} = b_{0j} = 0$$

for one value of j.

- The amplitude response of an arbitrary recursive filter can be deduced as

$$M(\mathbf{x}, \omega) = |H(e^{j\omega T})|$$

where ω is the frequency and

$$\mathbf{x} = [a_{01}\ a_{11}\ b_{01}\ b_{11}\ \cdots\ b_{1J}\ H_0]^T$$

is a column vector with $4J + 1$ elements.

Problem Formulation *Cont'd*

- An approximation error can be constructed as the difference between the actual amplitude response $M(\mathbf{x}, \omega)$ and the desired amplitude response $M_0(\omega)$ as

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- By sampling $e(\mathbf{x},\,\omega)$ at frequencies $\omega_1,\ \omega_2,\ \ldots,\ \omega_K$, the column vector

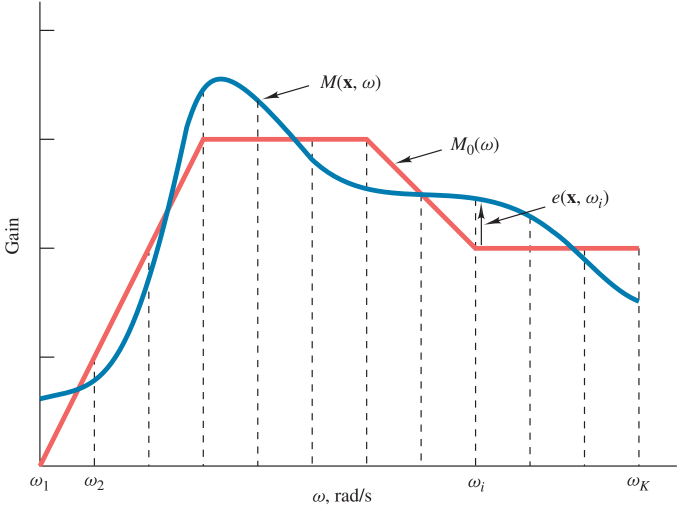
$$\mathbf{E}(\mathbf{x}) = [e_1(\mathbf{x})\ e_2(\mathbf{x})\ \ \ldots\ \ \ e_K(\mathbf{x})]^T$$

can be formed where

$$e_i(\mathbf{x}) = e(\mathbf{x},\,\omega_i)$$

for $i = 1, 2, \ldots, K$.

Problem Formulation *Cont'd*



- A recursive filter can be designed by finding a point $\mathbf{x} = \breve{\mathbf{x}}$ such that

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- Such a point can be obtained by minimizing the *L_p norm* of $\mathbf{E}(\mathbf{x})$, which is defined as

$$\Psi(\mathbf{x}) = L_p(\mathbf{x}) = ||\mathbf{E}(\mathbf{x})||_p = \left[\sum_{i=1}^{K} |e_i(\mathbf{x})|^p\right]^{1/p}$$

where p is a positive integer.

- For filter design, the most important norms are the L_2 and L_∞ norms which are defined as

$$L_2(\mathbf{x}) = \left[\sum_{i=1}^{K} |e_i(\mathbf{x})|^2\right]^{1/2}$$

and
$$\begin{aligned}L_\infty(\mathbf{x}) &= \lim_{p\to\infty}\left\{\sum_{i=1}^{K} |e_i(\mathbf{x})|^p\right\}^{1/p} \\ &= \widehat{E}(\mathbf{x}) \lim_{p\to\infty}\left\{\sum_{i=1}^{K}\left[\frac{|e_i(\mathbf{x})|}{\widehat{E}(\mathbf{x})}\right]^p\right\}^{1/p} \\ &= \widehat{E}(\mathbf{x})\end{aligned}$$

where $\widehat{E}(\mathbf{x}) = \underset{1\leq i\leq K}{max} \; |e_i(\mathbf{x})|$

- In summary, a recursive filter with an amplitude response that approaches a specified amplitude response $M_0(\omega)$ can be designed by solving the optimization problem

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- If

$$\Psi(\mathbf{x}) = L_2(\mathbf{x})$$

a *least-squares* solution is obtained and if

$$\Psi(\mathbf{x}) = L_\infty(\mathbf{x})$$

the outcome will be a so-called *minimax* solution.

Problem Formulation *Cont'd*

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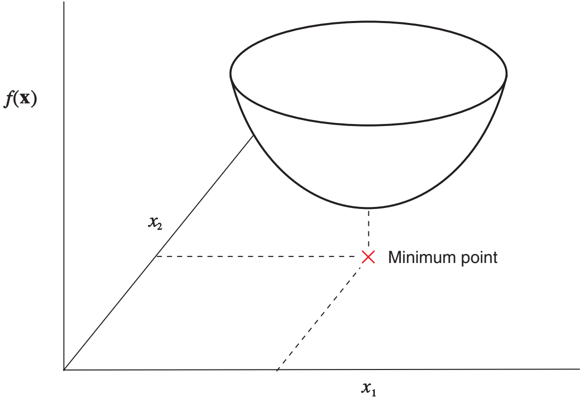
- The optimization problem obtained can be solved by using a great variety of *unconstrained* optimization algorithms that have evolved since the invention of computers.
- In the next series of slides, various implementations of the most fundamental unconstrained optimization algorithm, namely, the *Newton algorithm*, will be presented.
- In due course, the family of *quasi-Newton algorithms* will be described which, as may be expected, are based on the Newton algorithm.

  These algorithms have been found to be *quite robust and very efficient* in many applications including the design of recursive digital filters.

Newton Algorithm

- The Newton algorithm is based on Newton's classical method for finding the minimum of a quadratic *convex* function.



- Consider a function $f(\mathbf{x})$ of n variables, where $\mathbf{x} = [x_1 \; x_2 \; \cdots \; x_n]^T$ is a column vector, and let $\boldsymbol{\delta} = [\delta_1 \; \delta_2 \; \cdots \; \delta_n]^T$ be a change in $\mathbf{x}$.

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- If $f(\mathbf{x})$ has continuous second derivatives, its Taylor series at point $\mathbf{x} + \boldsymbol{\delta}$ is given by

$$f(\mathbf{x}+\boldsymbol{\delta}) = f(\mathbf{x}) + \sum_{i=1}^{n} \frac{\partial f(\mathbf{x})}{\partial x_i}\delta_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j}\delta_i\delta_j + o\left(||\boldsymbol{\delta}||_2^2\right)$$

where the remainder $o(||\boldsymbol{\delta}||_2^2)$ approaches zero faster than $||\boldsymbol{\delta}||_2^2$.

- If the remainder is negligible and a *stationary point* exists in the neighborhood of some point $\mathbf{x}$, it can be determined by differentiating $f(\mathbf{x}+\boldsymbol{\delta})$ with respect to elements δ_k for $k = 1, 2, \ldots, n$, and setting the result to zero, i.e.,

$$\frac{\partial f(\mathbf{x}+\boldsymbol{\delta})}{\partial \delta_k} = \frac{\partial f(\mathbf{x})}{\partial x_k} + \sum_{i=1}^{n} \frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_k}\delta_i = 0$$

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- The solution of the above equation can be expressed as

$$\boldsymbol{\delta} = -\mathbf{H}^{-1}\mathbf{g}$$

where $\mathbf{g}$ is the *gradient vector* and $\mathbf{H}$ is the *Hessian matrix*.

...

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- The gradient and Hessian are given by

$$\mathbf{g} = \nabla f(\mathbf{x}) = \left[\frac{\partial f(\mathbf{x})}{\partial x_1} \;\; \frac{\partial f(\mathbf{x})}{\partial x_2} \;\; \cdots \;\; \frac{\partial f(\mathbf{x})}{\partial x_n} \right]^T$$

and

$$\mathbf{H} = \begin{bmatrix} \frac{\partial^2 f(\mathbf{x})}{\partial x_1^2} & \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(\mathbf{x})}{\partial x_2 \partial x_1} & \frac{\partial^2 f(\mathbf{x})}{\partial x_2^2} & \cdots & \frac{\partial^2 f(\mathbf{x})}{\partial x_2 \partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 f(\mathbf{x})}{\partial x_n \partial x_1} & \frac{\partial^2 f(\mathbf{x})}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f(\mathbf{x})}{\partial x_n^2} \end{bmatrix}$$

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- If $f(\mathbf{x})$ is a *quadratic* function, its second partial derivatives are constants, i.e., $\mathbf{H}$ is a constant symmetric matrix, and its third and higher derivatives are zero, i.e., condition (1) holds.
- If $f(\mathbf{x})$ has a minimum at a stationary point, then the Hessian matrix is *positive definite* at the minimum point.

  In such a case, the Hessian is *nonsingular*, i.e., condition (2) holds.

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- When the two conditions apply, we go to MATLAB and very quickly we obtain the solution of the optimization problem, i.e., the minimum point and the minimum value of the function, are obtained as

$$\breve{\mathbf{x}} = \mathbf{x} + \breve{\boldsymbol{\delta}}, \quad f(\breve{\mathbf{x}})$$

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- No optimization algorithms are necessary.

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- If condition (2) is violated, the equation either has an infinite number of solutions or it has no solutions at all.

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- As a result, in the neighborhood of the solution, conditions (1) and (2) are again satisfied and the equation

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will yield an accurate estimate of the minimum point.

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- However, if we could find a way of getting near the solution, we would immediately be able to get the solution itself by evaluating
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- Thus in order to be able to construct an unconstrained optimization algorithm for the solution of nonquadratic problems, we need to find a mechanism that will enable the algorithm to get to the locale of the solution starting from an arbitrary initial point.

- A basic strategy adopted in most unconstrained optimization algorithms is to apply a series of corrections to an initial point ensuring that each correction is made in a direction that will *reduce* the objective function.

  Such directions are known as *descent directions*.

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- It turns out that if the Hessian $\mathbf{H}$ is positive definite, then so is the inverse Hessian $\mathbf{H}^{-1}$. In such a case, the direction vector

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- Since the above descent direction would give the solution in just one shot in a quadratic problem, it is commonly referred to as the *Newton direction*.

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  Nevertheless, it will reduce the value of the objective function by a certain amount and, therefore, a new point will be obtained that is closer to the solution.

- To maximize the reduction achieved in the objective function and thereby get closer to the solution, we let

$$\boldsymbol{\delta} = \alpha \mathbf{d}$$

  and choose parameter α such that the objective function, $f(\boldsymbol{x} + \alpha \mathbf{d})$, is minimized.

  We can do that by using a 1-dimensional optimization algorithm also known as a *line search*.

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- To circumvent this problem, we simply *force* the Hessian to become positive definite, for example, we could assign

$$\mathbf{H} = \mathbf{I}$$

  where $\mathbf{I}$ is the identity matrix.

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5. Set $\mathbf{x}_{k+1} = \mathbf{x}_k + \boldsymbol{\delta}_k$, where $\boldsymbol{\delta}_k = \alpha_k\mathbf{d}_k$, and compute $f_{k+1} = f(\mathbf{x}_{k+1})$.

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6. If $||\alpha_k\mathbf{d}_k||_2 < \varepsilon$, then output $\breve{\mathbf{x}} = \mathbf{x}_{k+1}$, $f(\breve{\mathbf{x}}) = f_{k+1}$, and stop.

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6. If $||\alpha_k\mathbf{d}_k||_2 < \varepsilon$, then output $\breve{\mathbf{x}} = \mathbf{x}_{k+1}$, $f(\breve{\mathbf{x}}) = f_{k+1}$, and stop.

   Otherwise, set $k = k + 1$ and repeat from step 2.

Newton Algorithm *Cont'd*

- In Step 4 of the algorithm, a *line search* is used to find the value of α_k that minimizes the value of the objective function, $f(\mathbf{x}_k + \alpha\,\mathbf{d}_k)$, along the Newton direction.

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- Any *1-dimensional optimization algorithm* can be used as a line search.Therefore, a great variety of possible line searches

  are available. See Chap. 4 of

  A. Antoniou and W.-S. Lu, *Practical Optimization: Algorithms and Engineering Applications*, Springer, 2007, at the following link:

  http://www.ece.uvic.ca/~optimize

- In Step 6, the algorithm is terminated if the L_2 norm of $\alpha_k \mathbf{d}_k$, i.e., the *magnitude of the change in* $\mathbf{x}$, is less than ε.

  The parameter ε is said to be the *termination tolerance* and it is a small positive constant whose value is determined by the application under consideration.

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- In certain applications, a termination tolerance on the *change in the objective function* itself, e.g., $|f_{k+1} - f_k| < \varepsilon$, may be preferable and sometimes termination tolerances may be imposed on the magnitudes of *both* the changes in $\mathbf{x}$ and the objective function.

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- As the solution is approached, the objective function behaves more and more like a quadratic function, parameter α assumes values that are closer and closer to unity, and the Newton directions give new points that are closer and closer to the solution.

Newton Algorithm *Cont'd*

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- As the solution is approached, the objective function behaves more and more like a quadratic function, parameter α assumes values that are closer and closer to unity, and the Newton directions give new points that are closer and closer to the solution.
- Eventually, a point is obtained that is sufficiently close to the solution and convergence is deemed to have been achieved.

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Newton Algorithm *Cont'd*

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- As the solution is approached, progress becomes very rapid and the algorithm appears to *zoom* to the solution.
- This feature is shared by all the algorithms based on Newton's classical method for finding the minima of a function, including the family of quasi-Newton algorithms.

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- The best we can hope for is a solution that satisfies the required specifications.
- This is usually achieved by using different initialization points.

This slide concludes the presentation.

Thank you for your attention.